Contributions to Graph Theory

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CONTRIBUTIONS TO GRAPH THEORY

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by

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Preface

In the name of Allaah, the Most Gracious and the Most Merciful.
All the praises and thanks are to Allaah, the Lord of the 'aalamiin (mankind, jinn and all that exists).

This thesis is the result of research between January 2002 and February 2005 in three topics of graph theory, namely: spanning 2-connected subgraphs of some classes of grid graphs, Ramsey numbers for paths versus other graphs, and λ-backbone colorings. The papers that together underlay this thesis are listed below.

Publications in refereed journals

1. A.N.M. Salman, E.T. Baskoro and H.J. Broersma, A note concerning Hamilton cycles in some classes of grid graphs, *Proceedings ITB Sains dan Teknologi* **35A** (1) (2003) 65–70.

2. A.N.M. Salman, E.T. Baskoro and H.J. Broersma, Spanning 2-connected subgraphs in truncated rectangular grid graphs, *Journal of Indonesian Mathematical Society* **8** (4) (2002) 63–68.

3. A.N.M. Salman, E.T. Baskoro and H.J. Broersma, Spanning 2-connected subgraphs in alphabet graphs, special classes of grid graphs, *Journal of Automata, Languages and Combinatorics* **8** (4) (2003) 675–681.

4. A.N.M. Salman, E.T. Baskoro, H.J. Broersma and C.A. Rodger, More on spanning 2-connected subgraphs in truncated rectangular grid graphs, *Bulletin of the Institute of Combinatorics and Its Applications* **39** (2003) 31–38.

5. A.N.M. Salman, H.J. Broersma and C.A. Rodger, More on spanning 2-connected subgraphs of alphabet graphs, special classes of grid graphs, *Accepted for publication in Bulletin of the Institute of Combinatorics and Its Applications* (2004).

6. A.N.M. Salman, H.J. Broersma and C.A. Rodger, A continuation of spanning 2-connected subgraphs in truncated rectangular grid graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing* **49** (2004) 177–186.

7. A.N.M. Salman and H.J. Broersma, Path-fan Ramsey numbers, *Accepted for publication in Discrete Applied Mathematics* (2004).

8. A.N.M. Salman and H.J. Broersma, On Ramsey numbers for paths versus wheels, *Accepted for publication in Discrete Mathematics* (2004).

9. A.N.M. Salman and H.J. Broersma, *Path-kipas Ramsey numbers*, Preprint (2004).

10. A.N.M. Salman, H.J. Broersma, J. Fujisawa, L. Marchal, D. Paulusma and K. Yoshimoto, *λ-Backbone colorings along pairwise disjoint stars and matchings*, Preprint (2004).

11. A.N.M. Salman, H.J. Broersma and D. Paulusma, *The computational complexity of λ-backbone coloring*, Preprint (2004).

12. H.J. Broersma, L. Marchal, D. Paulusma and A.N.M. Salman, *λ-Backbone coloring numbers of split graphs along trees, stars or matchings*, Preprint (2005).

Publications in conference proceedings

1. A.N.M. Salman, E.T. Baskoro and H.J. Broersma, *Spanning 2-connected subgraphs of alphabet graphs, special classes of grid graphs*, in: Proceedings of the Thirteenth Australasian Workshop on Combinatorial Algorithms (2002) 151–161.

2. A.N.M. Salman and H.J. Broersma, *The Ramsey numbers of paths versus fans*, in: Scientific Program of the 2nd Cologne Twente Workshop on Graphs and Combinatorial Optimization (2003) 106–110.

3. A.N.M. Salman and H.J. Broersma, *Some lower bounds and upper bounds for path-wheel Ramsey numbers*, in: Proceeding ISTECS (2003) 1–4.

4. A.N.M. Salman and H.J. Broersma, *The Ramsey numbers of paths versus kipases*, in: Scientific Program of CTW04 Workshop on Graphs and Combinatorial Optimization (2004) 218–222.

Presentations

1. Spanning 2-connected subgraphs in truncated rectangular grid graphs, *The First IAMS-N Seminar on Applied Mathematics*, Enschede, The Netherlands, May 23-24, 2002.

2. Spanning 2-connected subgraphs of alphabet graphs, special classes of grid graphs, *The Thirteenth Australasian Workshop on Combinatorial Algorithms*, Fraser Island, Australia, July 7-10, 2002.

3. Spanning 2-connected subgraphs in truncated rectangular grid graphs and some open problems, *Post Australasian Workshop on Combinatorial Algorithms*, Newcastle, Australia, July 13-14, 2002.

4. The Ramsey numbers of paths versus fans, *The 2nd Cologne Twente Workshop on Graphs and Combinatorial Optimization*, Enschede, The Netherlands, May 14-16, 2003.

5. Some lower bounds and upper bounds for path-fan Ramsey numbers, *One Day IAMS-N Seminar on Applied Mathematics*, Eindhoven, The Netherlands, May 27, 2003.

6. Path-wheel Ramsey numbers, *The 12th Workshop on Cycles and Colourings*, High Tatras, Slovakia, September 1-5, 2003.

7. Some lower bounds and upper bounds for path-wheel Ramsey numbers, *Indonesian Students' Scientific Meeting*, Delft, The Netherlands, October 9-10, 2003.

8. On Ramsey numbers $R(P_n, G)$, *EIDMA 2003 Symposium*, Mierlo, The Netherlands, November 13-14, 2003.

9. The Ramsey numbers of paths versus kipases, *CTW04 Workshop on Graphs and Combinatorial Optimization*, Menaggio, Italy, May 31-June 2, 2004.

10. The computational complexity of λ-backbone coloring, *EIDMA 2004 Symposium*, Mierlo, The Netherlands, November 25-26, 2004.

Only with God's blessing I got a lot of help, support and encouragement from many people so that I could finish writing this thesis. For that, I would like to sincerely give my praises to God and to express my gratitude to all the people who helped me in accomplishing this thesis.

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Enschede, April 2005

M. Salman A.N.

Contents

Chapter 1

General Introduction

Abstract In this chapter we present some notations and give a survey of the existing results about three topics of graph theory that are considered in this thesis, namely: spanning 2-connected subgraphs of grid graphs, Ramsey numbers for paths versus other graphs, and a general framework for coloring problems.

1.1 Notation and terminology

Throughout this thesis, we use [3] for terminology and notation not defined here and consider only finite and simple graphs. Let G be such a graph. We write $V(G)$ or V for the vertex set of G and $E(G)$ or E for the edge set of G. The graph $H = (V', E')$ is called a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$ (implying that the edges of H have all their end vertices in V').

If $e = \{u, v\} \in E$ (in short, $e = uv$), then u is called *adjacent* to v, and u and v are called *neighbors*. For $x \in V$, define $N(x) = \{y \in V | xy \in E\}$ and $N[x] = N(x) \cup \{x\}$. A *perfect matching* of G is a subset of $|V|/2$ edges of E that are pairwise vertex-disjoint.

If $S \subset V(G)$, $S \neq V(G)$, then $G - S$ denotes the subgraph of G induced by $V(G) \setminus S$. If $|S| = 1$, then we also use $G - z$ for $S = \{z\}$ instead of $G - \{z\}$. If $e \in E(G)$, then $G - e = (V(G), E(G) \setminus \{e\})$. A set $V' \subseteq V$ is called *independent* if G does not contain edges with both end vertices in V'.

A *path* is a graph P whose vertices can be ordered into a sequence $v_1, v_2, \ldots, v_n$ such that $E_P = \{v_1v_2, \ldots, v_{n-1}v_n\}$. A *Hamilton path* of the graph G is a path containing all vertices of G. The *distance* between two vertices u and v of a connected graph is the length of a shortest path between them. A *cycle* is a graph C whose vertices can be ordered into a sequence $v_1, v_2, \ldots, v_n$ such that $E_C = \{v_1v_2, \ldots, v_{n-1}v_n, v_nv_1\}$. A *tree* is a connected graph T that does not contain any cycles. We denote the path, the cycle and the tree on n vertices by P_n, C_n and T_n, respectively.

A *complete graph* is a graph with an edge between every pair of vertices. The complete graph on n vertices is denoted by K_n. The graph $\overline{G}$ is the *complement* of G, i.e., the graph obtained from the complete graph on $|V(G)|$ vertices by deleting the edges of G.

A graph G is *complete p-partite* if its vertices can be partitioned into p non-empty independent sets $V_1, \ldots, V_p$ such that its edge set E is formed by all edges that have one end vertex in V_i and the other one in V_j for some $1 \leq i < j \leq p$. A complete 2-partite graph is called a *complete m by n bipartite graph* and denoted by $K_{m,n}$ if $|V_1| = m$ and $|V_2| = n$. A *star* S_n is a complete 2-partite graph with independent sets $V_1 = \{r\}$ and V_2 with $|V_2| = n$; the vertex r is called the *root* and the vertices in V_2 are called the *leaves* of S_n.

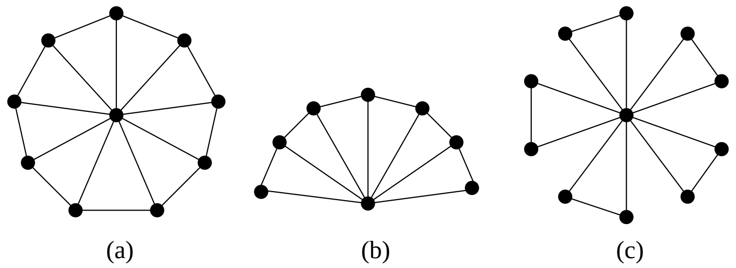


Figure 1.1: (a) The wheel W_9 (b) The kipas $\hat{K}_7$ (c) The fan F_5

A *wheel* W_m is a graph on $m+1$ vertices obtained from a cycle on m vertices by adding a new vertex and edges joining it to all the vertices of the cycle (W_m is the join of K_1 and C_m). A *kipas* $\hat{K}_m$ is a graph on $m+1$ vertices obtained from the join of K_1 and P_m. A *fan* F_m is a graph on $2m+1$ vertices obtained from m disjoint triangles (K_3s) by identifying precisely one vertex of every triangle (F_m is the join of K_1 and mK_2). It is also known in the literature as 'dutch windmill'. For illustration, consider W_9 in Figure 1.1(a), $\hat{K}_7$ in Figure 1.1(b), and F_5 in Figure 1.1(c). The vertex corresponding to K_1 in a wheel or in a kipas or in a fan is called the *hub* of the wheel or the *hub* of the kipas or the *hub* of the fan, respectively.

A *split graph* is a graph whose vertex set can be partitioned into a *clique* (i.e. a set of mutually adjacent vertices) and an *independent set* (i.e. a set of mutually nonadjacent vertices), with possibly edges in between. The size of a largest clique in G and the size of a largest independent set in G are denoted by $\omega(G)$ and $\alpha(G)$, respectively.

Let $G = (V, E)$ be a graph. A *vertex coloring* $f : V \to \{1, 2, 3, \ldots\}$ of V is *proper*, if $|f(u) - f(v)| \geq 1$ holds for all edges $uv \in E$. A proper vertex coloring $f : V \to \{1, \ldots, k\}$ is called a *k-coloring*, and the *chromatic number* $\chi(G)$ is the smallest integer k for which there exists a k-coloring. By definition, a k-coloring partitions V into k independent sets $V_1, \ldots, V_k$.

1.2 Spanning 2-connected subgraphs of grid graphs

A subgraph H of a graph $G = (V, E)$ is called a *spanning subgraph* if $V(H) = V$. A connected graph is called 2-*connected* if it remains connected if at most one vertex is removed. A *Hamilton cycle* in a graph $G = (V, E)$ is a cycle containing every vertex of V, i.e. a spanning 2-connected subgraph in which every vertex has degree 2 (the number of edges is $|V|$).

It is probable that no efficient algorithm exists for finding Hamilton cycles, but that does not prevent the problem from arising in real applications. There are a number of ways to cope with this dilemma. One might be satisfied with an approximation - for example, a cycle that covers most but not all of the vertices of the graph. Or, the particular instance of the problem might be a special case that is solvable efficiently - for example, complete graphs always have a Hamilton cycle, and it is very easy to find. Finally, if an exact solution is required, the inefficient enumerative algorithm (or variant thereof) might be tried with the hope that its actual performance on this particular instance of the problem does not approach the worst case. The search for restricted cases is then of obvious relevance to the second option, and quite possibly a useful starting point for certain approximations if the first option is pursued.

The *infinite grid graph* G^∞ is defined by the set of vertices $V = \{(x, y) \mid x \in Z,\ y \in Z\ \}$ and the set of edges E between all pairs of vertices from

V at Euclidean distance precisely 1. For any integers $s \geq 1$ and $t \geq 1$, the *rectangular grid graph* $R(s,t)$ is the (finite) subgraph of G^∞ induced by $V(s,t) = \{(x,y) \mid 1 \leq x \leq s, 1 \leq y \leq t, x \in Z, y \in Z\}$ (and just containing all edges from G^∞ between pairs of vertices from $V(s,t)$). This graph $R(s,t)$ is also known as the *product graph* $P_s \times P_t$ of two disjoint paths P_s and P_t. A *grid graph* is a graph that is isomorphic to a subgraph of $R(s,t)$ induced by a subset of $V(s,t)$ for some integers $s \geq 1$ and $t \geq 1$.

Grid graphs have three important properties that in many cases permit efficient algorithms for a variety of graph problems. The first is that grid graphs are *planar graph*, i.e. they can be drawn in the plane $\Re^2$ in such a way that the edges only intersect at the vertices of the graph. In such a drawing for the grid graph $G = (V,E)$, the regions of $\Re^2 \backslash (V \cup E)$ are called the *faces* of G. Exactly one of the faces is unbounded; this is called the *outer face*; the others are its *inner faces*. The *natural drawing* of a grid graph is just described by drawing its vertices in $\Re^2$ according to their coordinates. A *solid grid graph* is a grid graph all of whose inner faces have area one (are bounded by a cycle on four vertices) in a natural drawing. A grid graph that is not solid contains inner faces (in a natural drawing) that have area larger than one; these faces are called *holes*. The second property of grid graphs is that they are *bipartite*, which means that the vertices of the graph can be partitioned into two sets so that all edges have one end vertex in each set. Finally, the maximum degree of all vertices is four. Unfortunately, for the Hamilton cycle problem, these features are not likely to simplify the problem enough to permit an efficient algorithm.

Itai, Papadimitriou and Szwarcfiter [29] proved that deciding whether a given grid graph has a Hamilton cycle is an NP-complete problem. This implies that the problem of finding a spanning 2-connected subgraph with as few edges as possible is also NP-hard for grid graphs. It has been conjectured that the first problem remains NP-complete when it is restricted to solid grid graphs. However, Umans and Lenhart [50] recently proved that this problem is polynomially solvable, by presenting a complicated algorithm with time complexity $O(|V|^4)$. In a recent paper of Sheffield [48] the work of [29] has been extended to grid graphs with a small number of holes. For the second problem the complexity is not known when it is restricted to solid grid graphs. It remains an open problem –what the complexity of both problems is –when we restrict ourselves to grid graphs with a fixed number of holes.

Motivated by the above problems, we studied the problem of the existence of a Hamilton cycle and the problem of determining a spanning 2-connected subgraph with as few edges as possible for some classes of finite grid graphs with no or a few holes. We define four classes of grid graphs called truncated rectangular grid graphs and 26 classes of grid graphs called alphabet graphs. The definition of the classes can be found in Chapter 2. We present our results from [38], [39], [40], [41] and [42] in the following theorems.

Theorem 1.2.1. *Let* $R(s,t)^{-1(k,l)}$, $R(s,t)^{-2(k,l)}$, $R(s,t)^{-3(k,l)}$ *and* $R(s,t)^{-4(k,l)}$ *denote the* 1*-corner truncated rectangular grid graph, the* 2*-corner truncated rectangular grid graph, the* 3*-corner truncated rectangular grid graph and the* 4*-corner truncated rectangular grid graph, respectively. Then:*

(a) $R(s,t)^{-1(k,l)}$ *contains a spanning* 2*-connected subgraph with (at most)* $|V|+1$ *edges and is hamiltonian if and only if both* $s \cdot t$ *and* $k \cdot l$ *are even or both* $s \cdot t$ *and* $k \cdot l$ *are odd.*

(b) $R(s,t)^{-2(k,l)}$ *contains a spanning* 2*-connected subgraph with*
• $|V|$ *edges if* $s \cdot t$ *is even and at least one of* k *and* l *is even if both* s *and* t *are even;*
• $|V|+2$ *edges if* s *and* t *are even and* k *and* l *are odd;*
• $|V|+1$ *edges in all other cases.*
These numbers of edges are all best possible.

(c) $R(s,t)^{-3(k,l)}$ *contains a spanning* 2*-connected subgraph with*
• $|V|$ *edges if both* $s \cdot t$ *and* $k \cdot l$ *are even;*
• $|V|+2$ *edges if all of* s, t, k *and* l *are odd;*
• $|V|+1$ *edges in all other cases.*
These numbers of edges are all best possible.

(d) $R(s,t)^{-4(k,l)}$ *contains a spanning* 2*-connected subgraph with (at most)* $|V|+3$ *edges and is hamiltonian if and only if* $s \cdot t$ *is even. The bound* $|V|+3$ *is best possible for any odd numbers* s, t, k *and* l.

Theorem 1.2.2. *Let* $m \geq 3$ *and* $n \geq 3$. *Let* $A, B, \ldots, Z$ *denote the alphabet graphs* $A_{m,n}$, $B_{m,n}$, $\ldots$, $Z_{m,n}$. *Then:*

(a) A, D, O *and* P *are hamiltonian.*

(b) E *and* F *contain a spanning* 2*-connected subgraph with (at most)* $|V|+1$ *edges and are hamiltonian if and only if* n *is even.*

(c) N *contains a spanning* 2*-connected subgraph with (at most)* $|V|+1$ *edges and is hamiltonian if and only if* m *and* n *have a different parity.*

(d) Q contains a spanning 2-connected subgraph with (at most) $|V|+1$ edges and is hamiltonian if and only if m is odd or n is even.

(e) R contains a spanning 2-connected subgraph with (at most) $|V|+1$ edges and is hamiltonian if and only if m is even or n is odd.

(f) W contains a spanning 2-connected subgraph with
- *$|V|$ edges if m is even;*
- *$|V|+1$ edges if both m and n are odd;*
- *$|V|+2$ edges if m is odd and n is even.*

These numbers of edges are all best possible.

(g) X contains a spanning 2-connected subgraph with
- *$|V|$ edges if either (m is even) or (m is odd, $m \geq 7$ and n is even);*
- *$|V|+1$ edges if either (m and n are odd) or (m=5 and n is even);*
- *$|V|+2$ edges if m=3 and n is even.*

(h) The remaining alphabet graphs contain a spanning 2-connected subgraph with (at most) $|V|+1$ edges and are hamiltonian if and only if $m \cdot n$ is even.

1.3 Ramsey numbers for paths versus other graphs

Generalized Ramsey numbers have received a great deal of attention over the last several years [35]. In this section we consider the Ramsey numbers for paths versus other graphs.

For two given graphs F and H, the *Ramsey number* $R(F, H)$ is the smallest positive integer p such that for every graph G on p vertices the following holds: either G contains F as a subgraph or the complement of G contains H as a subgraph.

The definition of the Ramsey number $R(F, H)$ evidently first appeared in a paper of Geréncser and Gyárfás which dealt with the case where F and H are both paths. Their result is rewritten in Theorem 1.3.1.

Theorem 1.3.1. (Geréncser & Gyárfás [20])

$$R(P_n, P_m) = m + \left\lfloor \frac{n}{2} \right\rfloor - 1 \quad \textit{for } 2 \leq n \leq m.$$

After that, the Ramsey numbers $R(P_n, G)$ for paths versus other graphs G have been investigated in several papers.

In 1973 Parsons found the Ramsey numbers for paths versus complete graphs which are formulated in Theorem 1.3.2.

Theorem 1.3.2. (Parsons [33])

$$R(P_n, K_m) = (m-1)(n-1) + 1.$$

Faudree, Lawrence, Parsons and Schelp determined the Ramsey numbers for paths versus cycles in 1974.

Theorem 1.3.3. (Faudree, Lawrence, Parsons & Schelp [13])

$$R(P_n, C_m) = \begin{cases} 2n-1 & for\ 3 \leq odd\ m \leq n \\ n + \frac{m}{2} - 1 & for\ 4 \leq even\ m \leq n \\ max\left\{m + \left\lfloor \frac{n}{2} \right\rfloor - 1, 2n-1\right\} & for\ 2 \leq n \leq odd\ m \\ m + \left\lfloor \frac{n}{2} \right\rfloor - 1 & for\ 2 \leq n \leq even\ m. \end{cases}$$

The Ramsey numbers $R(P_n, S_m)$ for paths versus stars for all m and n were given by Parsons in 1974. He presented the numbers by explicit formulas as in Theorem 1.3.4 and Theorem 1.3.5, and by a recurrence as in Theorem 1.3.6.

Theorem 1.3.4. (Parsons [34])

$$R(P_n, S_m) = \begin{cases} 1 & for\ n = 1 \\ m+n-1 & for\ n \geq 2,\ m = 1\ mod(n-1) \\ 2m-1 & for\ n \geq 3,\ m+1 \leq n \leq 2m-1 \\ n & for\ m \geq 2,\ n \geq 2m. \end{cases}$$

Theorem 1.3.5. (Parsons [34])
If $(n \geq 3,\ m > n,\ m \geq (n-3)^2$ *and* $m \neq 1\ mod(n-1))$ *or* $(n \geq 6,$ $n < m < (n-3)^2$ *and* $m = 2\ mod(n-1))$ *or* $(n \geq 7,\ n < m < (n-3)^2$ *and* $m = 0\ mod(n-1))$ *or* $(n \geq 7,\ n < m < (n-3)^2$ *and* $m = -1\ mod(n-1))$ *or* $(n \geq 7,\ n < m < (n-3)^2,\ m \neq 1\ mod(n-1)$ *and* $m = 1\ mod(n-2))$, *then*

$$R(P_n, S_m) = m + n - 2.$$

Theorem 1.3.6. (Parsons [34])

$$R(P_n, S_m) = max\{R(P_{n-1}, S_m), R(P_n, S_{m-n+1}) + n - 1\} \quad for\ 3 \leq n \leq m.$$

In 1978 Rousseau and Sheehan gave the Ramsey numbers $R(P_n, K_l + \overline{K}_m)$, where $K_l + \overline{K}_m$ denotes the joint of the complete graph on l vertices and the empty graph on m vertices.

Theorem 1.3.7. (Rousseau & Sheehan [36])
If $l \geq 1$, $m \geq 1$ and $n \geq 2$, then

$$R(P_n, K_l + \overline{K}_m) =$$
$$1 + max\left\{\left(\left\lfloor\frac{m-1}{n-1}\right\rfloor + l\right)(n-1),\ m - 1 + \left\lfloor\frac{m-1}{\lfloor(m-1)/(n-1)\rfloor + 1}\right\rfloor l\right\}.$$

Burr, Erdös, Faudree, Rousseau and Schelp determined the Ramsey numbers for paths versus sparse graphs in 1982 as the next theorem.

Theorem 1.3.8. (Burr, Erdös, Faudree, Rousseau & Schelp [9])
Let G be a connected graph with k vertices and no more than $\lceil k(1 + 1/81n^5)\rceil$ edges. If $\Delta(G) \leq k(1 - 1/81n^5)$, $k \geq 352n^{12}$ and $n \geq 2$, then

$$R(P_n, G) = k + \lceil n/2\rceil - 1.$$

In 1989 Häggkvist gave upper bounds for the path-complete bipartite Ramsey numbers as Theorem 1.3.9 and the exact values for a special case as in Theorem 1.3.10.

Theorem 1.3.9. (Häggkvist [24])

$$R(P_n, K_{q,m}) \leq q + m + n - 2.$$

Theorem 1.3.10. (Häggkvist [24])

$$R(P_n, K_{q,m}) = q + m + n - 2 \quad for\ q = 1\ mod(n-1)\ and\ m = 1\ mod(n-1).$$

Some upper bounds for the path-tree Ramsey numbers were given by Faudree, Schelp and Simonovits in 1990 as follows.

Theorem 1.3.11. (Faudree, Schelp & Simonovits [14])

$$R(P_n, T_m) \leq \begin{cases} m+n-2 & \text{for } n \geq m \text{ or } m \geq 432n^6 - n^2 \\ m+6n^2-2n & \text{for other values of } m \text{ and } n. \end{cases}$$

Now we consider the path-wheel Ramsey numbers. In 2001 Surahmat and Baskoro studied the Ramsey numbers for paths versus W_4 or W_5. Their result is rewritten in Theorem 1.3.12.

Theorem 1.3.12. (Surahmat & Baskoro [49])
Let $n \geq 3$. Then

$$R(P_n, W_m) = \begin{cases} 2n-1 & \text{for } m = 4 \\ 3n-2 & \text{for } m = 5. \end{cases}$$

In 2002 Chen, Zhang and Zhang obtained the path-wheel Ramsey numbers for the values of m and n that are presented in Theorem 1.3.13.

Theorem 1.3.13. (Chen, Zhang and Zhang [12])
Let $n \geq m-1$. Then

$$R(P_n, W_m) = \begin{cases} 2n-1 & \text{for even } m \geq 6 \\ 3n-2 & \text{for odd } m \geq 7. \end{cases}$$

In [44] we presented results which generalized the results in [49] and [12]. Those results are formulated in the following two theorems. The Ramsey numbers for 'small' paths versus wheels or paths versus 'small' wheels are presented in Theorem 1.3.14, and the Ramsey numbers for odd paths versus 'large' wheels are presented in Theorem 1.3.15. Moreover, we give lower bounds and upper bounds for $R(P_n, W_m)$ for other values of n and m as in Theorem 1.3.16 and Theorem 1.3.17.

Theorem 1.3.14.

$$R(P_n, W_m) = \begin{cases} 1 & \text{for } n = 1 \text{ and } m \geq 3 \\ m+1 & \text{for either } n = 2 \text{ and } m \geq 3 \\ & \text{or } n = 3 \text{ and even } m \geq 4 \\ m+2 & \text{for } n = 3 \text{ and odd } m \geq 5 \\ 3n-2 & \text{for either } n = 3 \text{ and } m = 3 \\ & \text{or } n \geq 4 \text{ and } 3 \leq \text{odd } m \leq 2n-1 \\ 2n-1 & \text{for } n \geq 4 \text{ and } 4 \leq \text{even } m \leq n+1. \end{cases}$$

Theorem 1.3.15. *If* $(n = 5$ *and* $m = 8$ *or* $m \geq 10)$ *or* (*odd* $n \geq 7$ *and* $m = 2n - 2$ *or* $m = 2n$ *or* $m \geq (n-3)^2$) *or* (*odd* $n \geq 9$ *and* $q \cdot n - 2q + 1 \leq m \leq q \cdot n - q + 2$ *with* $3 \leq q \leq n - 5$), *then*

$$R(P_n, W_m) = \begin{cases} m + n - 1 & \text{for } m = 1 \ mod(n-1) \\ m + n - 2 & \text{for other values of } m. \end{cases}$$

Theorem 1.3.16. *If* n *is odd,* $n \geq 7$ *and* $q \cdot n - q + 3 \leq m \leq q \cdot n - 2q + n - 2$ *with* $2 \leq q \leq n - 5$, *then*

$$m + n - 2 \geq R(P_n, W_m) \geq$$

$$max \left\{ \left\lfloor \frac{m}{n-1} \right\rfloor (n-1) + n, \ m + \left\lfloor \frac{m-1}{\lceil m/(n-1) \rceil} \right\rfloor \right\}.$$

Theorem 1.3.17. *If* ($n \geq 6$ *and* m *is even,* $n+2 \leq m \leq 2n-4$) *or* (n *is even,* $n \geq 4$ *and* $m = 2n - 2$ *or* $m \geq 2n$), *then*

$$m + \lfloor 3n/2 \rfloor - 2 \geq R(P_n, W_m) \geq$$

$$max \left\{ \left\lfloor \frac{m-1}{n-1} \right\rfloor (n-1) + n, \ m + \left\lfloor \frac{m-1}{\lceil (m-1)/(n-1) \rceil} \right\rfloor \right\}.$$

Next, we consider the path-kipas Ramsey numbers. In [45] we determined the Ramsey numbers $R(P_n, \hat{K}_m)$ for some values of n and m as in the following three theorems. Besides that, in Theorem 1.3.20, Theorem 1.3.21 and Theorem 1.3.22 we give lower bounds and upper bounds for $R(P_n, \hat{K}_m)$ for other values of m and n.

Theorem 1.3.18.

$$R(P_n, \hat{K}_m) = \begin{cases} 1 & \text{for } n = 1 \text{ and } m \geq 3 \\ m + 1 & \text{for either } n = 2 \text{ and } m \geq 3 \\ & \text{or } n = 3 \text{ and even } m \geq 4 \\ m + 2 & \text{for } n = 3 \text{ and odd } m \geq 5 \\ 3n - 2 & \text{for either } n = 3 \text{ and } m = 3 \\ & \text{or } n \geq 4 \text{ and } 3 \leq \text{odd } m \leq 2n - 1 \\ 2n - 1 & \text{for } n \geq 4 \text{ and } 4 \leq \text{even } m \leq n + 1. \end{cases}$$

Theorem 1.3.19. *If* $(4 \leq n \leq 6$ *and* $m = 2n-2$ *or* $m \geq 2n)$ *or* $(n \geq 7$ *and* $m = 2n-2$ *or* $m = 2n$ *or* $m \geq (n-3)^2)$ *or* $(n \geq 8$ *and* $q \cdot n - 2q + 1 \leq m \leq q \cdot n - q + 2$ *with* $3 \leq q \leq n-5)$, *then*

$$R(P_n, \hat{K}_m) = \begin{cases} m+n-1 & \text{for } m = 1 \text{ mod}(n-1) \\ m+n-2 & \text{for other values of } m. \end{cases}$$

Theorem 1.3.20. *If* n *is odd,* $n \geq 11$ *and* $q \cdot n - q + 3 \leq m \leq q \cdot n - 3q + n - 3$ *with* $2 \leq q \leq (n-7)/2$, *then*

$$m+n-3 \geq R(P_n, \hat{K}_m) \geq$$
$$max\left\{ \left\lfloor \frac{m}{n-1} \right\rfloor (n-1) + n,\ m + \left\lfloor \frac{m-1}{\lceil m/(n-1) \rceil} \right\rfloor \right\}.$$

Theorem 1.3.21. *If* n *is even,* $n \geq 8$ *and* $q \cdot n - q + 3 \leq m \leq q \cdot n - 2q + n - 2$ *with* $2 \leq q \leq n-5$, *then*

$$m+n-2 \geq R(P_n, \hat{K}_m) \geq$$
$$max\left\{ \left\lfloor \frac{m}{n-1} \right\rfloor (n-1) + n,\ m + \left\lfloor \frac{m-1}{\lceil m/(n-1) \rceil} \right\rfloor \right\}.$$

Theorem 1.3.22. *If* $n \geq 6$ *and* m *is even with* $n+2 \leq m \leq 2n-4$, *then*

$$m + \left\lfloor \frac{3n}{2} \right\rfloor - 2 \geq R(P_n, \hat{K}_m) \geq \begin{cases} 2n-1 & \text{for } n+2 \leq m \leq n + \lfloor n/3 \rfloor \\ \frac{3m}{2} - 1 & \text{for } n + \lfloor n/3 \rfloor < m \leq 2n-4. \end{cases}$$

In the last part of this section we present our results about the path-fan Ramsey numbers [43]. The Ramsey numbers for 'small' paths versus fans or paths versus 'small' fans are presented in Theorem 1.3.23. In Theorem 1.3.24 and Theorem 1.3.25 we present the Ramsey numbers for paths versus 'large' fans. Moreover, we also give lower bounds and upper bounds for $R(P_n, F_m)$ for other values of m and n.

Theorem 1.3.23.

$$R(P_n, F_m) = \begin{cases} 1 & \text{for } n = 1 \text{ and } m \geq 2 \\ 2m+1 & \text{for } n = 2 \text{ or } n = 3 \text{ and } m \geq 2 \\ 2n-1 & \text{for } n \geq 4 \text{ and } 2 \leq m \leq (n+1)/2. \end{cases}$$

Theorem 1.3.24. *If* $(4 \le n \le 6$ *and* $m \ge n-1)$ *or* $(n \ge 7$ *and* $m = n-1$ *or* $m = n$ *or* $m \ge (n-3)^2/2)$ *or* $(n \ge 8$ *and* $(q \cdot n - 2q + 1)/2 \le m \le (q \cdot n - q + 2)/2$ *with* $3 \le q \le n-5)$*, then*

$$R(P_n, F_m) = \begin{cases} 2m + n - 1 & \text{for } 2m = 1 \ mod(n-1) \\ 2m + n - 2 & \text{for other values of } m. \end{cases}$$

Theorem 1.3.25. *If* n *is odd,* $n \ge 9$ *and either* $((q \cdot n - 3q + 1)/2 \le m \le (q \cdot n - 2q)/2$ *with* $3 \le q \le (n-3)/2)$ *or* $((q \cdot n - q - n + 4)/2 \le m \le (q \cdot n - 2q)/2$ *with* $(n-1)/2 \le q \le n-5)$*, then* $R(P_n, F_m) = 2m + n - 3$.

Theorem 1.3.26. *If* n *is odd,* $n \ge 11$ *and* $(q \cdot n - q + 4)/2 \le m \le (q \cdot n - 3q + n - 3)/2$ *with* $2 \le q \le (n-7)/2$*, then*

$$2m + n - 3 \ge R(P_n, F_m) \ge$$

$$max\left\{ \left\lfloor \frac{2m}{n-1} \right\rfloor (n-1) + n,\ 2m + \left\lfloor \frac{2m-1}{\lceil 2m/(n-1) \rceil} \right\rfloor \right\}.$$

Theorem 1.3.27. *If* n *is even,* $n \ge 8$ *and* $(q \cdot n - q + 3)/2 \le m \le (q \cdot n - 2q + n - 2)/2$ *with* $2 \le q \le n-5$*, then*

$$2m + n - 2 \ge R(P_n, F_m) \ge$$

$$max\left\{ \left\lfloor \frac{2m}{n-1} \right\rfloor (n-1) + n,\ 2m + \left\lfloor \frac{2m-1}{\lceil 2m/(n-1) \rceil} \right\rfloor \right\}.$$

Theorem 1.3.28. *If* $n \ge 6$ *and* $(n+2)/2 \le m \le n-2$*, then*

$$2m + \left\lfloor \frac{3n}{2} \right\rfloor - 2 \ge R(P_n, F_m) \ge \begin{cases} 2n - 1 & \text{for } \frac{n+2}{2} \le m \le \frac{n + \lfloor n/3 \rfloor}{2} \\ 3m - 1 & \text{for } \frac{n + \lfloor n/3 \rfloor}{2} < m \le n-2. \end{cases}$$

1.4 A general framework for coloring problems

In the application area of frequency assignment graphs are used to model the topology and mutual interference between transmitters: the vertices of the graph represent the transmitters; two vertices are adjacent in the graph if the corresponding transmitters are so close (or so strong) that they are likely to interfere if they broadcast on the same or 'similar' frequency channels. The problem in practice is to assign a limited number of frequency channels in an economical way to the transmitters in such a way that interference is kept at an 'acceptable level'. This has led to various different types of coloring problems in graphs, depending on different ways to model the level of interference, the notion of similar frequency channels, and the definition of acceptable level of interference (See e.g. [25], [31]).

In [7] an attempt was made to capture a number of different coloring problems in a unifying model. This general framework is as follows:

> Given two graphs G_1 and G_2 with the property that G_1 is a (spanning) subgraph of G_2, one considers the following type of coloring problems: Determine a coloring of (G_1 and) G_2 that satisfies certain restrictions of type 1 in G_1, and restrictions of type 2 in G_2, using a limited number of colors.

Many known coloring problems related to frequency assignment fit into this general framework [6]. We mention some of them here explicitly.

First of all suppose that $G_2 = G_1^2$, i.e. G_2 is obtained from G_1 by adding edges between all pairs of vertices that are at distance 2 in G_1. If one just asks for a proper vertex coloring of G_2 (and G_1), this is known as the distance-2 coloring problem. So, a *distant*-2 *coloring* of a graph G is a coloring of the vertices of G such that vertices at distance one or two have different colors. The least number for which a distant-2 coloring exists is called the *distant*-2 *chromatic number* of G, denoted by $\chi_2(G)$. Much of the research has been concentrated on the case that G_1 is a planar graph and on the problem to find the relation between distant-2 chromatic number and maximum degree of the graph (see e.g. [1], [4], [5], [28], and [51]). In 2001 Molloy and Salavativour proved the following thoerem.

Theorem 1.4.1. (Molloy and Salavativour [32])
If G is a planar graph with maximum degree Δ, then

$$\chi_2 \leq \begin{cases} \left\lfloor \frac{5}{3}\Delta \right\rfloor + 78 \\ \left\lfloor \frac{5}{3}\Delta \right\rfloor + 24, & \textit{if } \Delta \geq 241. \end{cases}$$

In some versions of this problem one puts the additional restriction on G_1 that the colors should be sufficiently separated, in order to model practical frequency assignment problems in which interference should be kept at an acceptable level. One way to model this is to use positive integers for the colors (modeling certain frequency channels) and to ask for a coloring of G_1 and G_2 such that the colors on adjacent vertices in G_2 are different, whereas they differ by at least 2 on adjacent vertices in G_1. This problem is known as the radio coloring problem. So, a *radio coloring* of graph $G = (V, E)$ is a function $f : V \to N^+$ such that $|f(u) - f(v)| \geq 2$ if $uv \in E$ and $|f(u) - f(v)| \geq 1$ if the distance between u and v in G is 2. The notion of radio coloring was introduced by Griggs and Yeh [22] under the name $L(2,1)$-labeling. The *span* of radio coloring f of G is $max_{v \in V} f(v)$. The problem of determining a radio coloring with minimum span and the problem of determining the complexity of a radio coloring for some classes of graphs have received a lot of attention (see e.g. [2], [10], [15], [16], [17], [18], and [30].

The so-called radio labeling problem models a practical setting in which all assigned frequency channels should be distinct, with the additional restriction that adjacent transmitters should use sufficiently separated frequency channels. So, a *radio labeling* of graph $G = (V, E)$ is an injective function $f : V \to N^+$ such that $|f(u) - f(v)| \geq 2$ if $uv \in E$. Within the above framework this can be modeled by considering the graph G_1 that models the adjacencies of n transmitters, and taking $G_2 = K_n$, the complete graph on n vertices. The restrictions are clear: one asks for a proper vertex coloring of G_2 such that adjacent vertices in G_1 receive colors that differ by at least 2. We refer to [22] and [27] for more particulars.

In the last part of this section we model the situation that the transmitters form a network in which a certain substructure of adjacent transmitters (called the backbone) is more crucial for the communication than the rest of the network. This means we should put more restrictions on the assignment of frequency channels along the backbone than on the assignment of frequency channels to other adjacent transmitters. The backbone could e.g. model so-called hot spots in the network where a very busy pattern of communications takes place, whereas the other adjacent transmitters supply a more moderate service. We consider the problem of coloring the graph G_2 (that models the whole network) with a proper vertex coloring such that the colors on adjacent vertices in G_1 (that model the backbone) differ by at least $\lambda \geq 2$. So, for a spanning subgraph $H = (V, E_H)$ of $G = (V, E)$, a proper vertex coloring f of V is a λ-*backbone coloring* of (G, H), if $|f(u) - f(v)| \geq \lambda$ holds for all edges $uv \in$

E_H. The *λ-backbone coloring number* $\text{BBC}_\lambda(G,H)$ of (G,H) is the smallest integer ℓ for which there exists a λ-backbone coloring $f : V \to \{1,\dots,\ell\}$. Note that the notion of λ-backbone coloring was introduced in [7]. It in fact generalizes both radio coloring and radio labeling: radio coloring is the special case of 2-backbone coloring in which G_1 is the backbone of $G_2 = G_1^2$, while radio labeling is the special case in which G_1 is the backbone of K_n.

We call a spanning subgraph H of a graph G

- a *tree backbone* of G if H is a (spanning) tree;
- a *path backbone* of G if H is a (Hamilton) path;
- a *star backbone* of G if H is a collection of pairwise disjoint stars;
- a *matching backbone* of G if H is a perfect matching.

Obviously, $\text{BBC}_\lambda(G,H) \geq \chi(G)$ holds for any backbone H of a graph G. In order to analyze the maximum difference between these two numbers the following values can be introduced.

$\mathcal{T}_\lambda(k) = \max\{\text{BBC}_\lambda(G,T) \mid T \text{ is a tree backbone of } G, \text{ and } \chi(G)=k\}$;
$\mathcal{P}_\lambda(k) = \max\{\text{BBC}_\lambda(G,P) \mid P \text{ is a path backbone of } G, \text{ and } \chi(G)=k\}$;
$\mathcal{S}_\lambda(k) = \max\{\text{BBC}_\lambda(G,S) \mid S \text{ is a star backbone of } G, \text{ and } \chi(G)=k\}$;
$\mathcal{M}_\lambda(k) = \max\{\text{BBC}_\lambda(G,M) \mid M \text{ is a matching backbone of } G, \text{ and } \chi(G)=k\}$.

In 2003 Broersma, Fomin, Golovach and Woeginger [7] considered cases where the backbone is a spanning tree or a Hamilton path. In 2004 we considered cases where the backbone is a collection of pairwise disjoint stars or a perfect matching [46]. In [7] and [46] combinatorial and algorithmic aspects are treated. In [8] we considered the backbone coloring numbers of split graphs with star, matching or tree backbones. We consider algorithmic aspects for tree or path backbones in [47].

We summarize the main results from [7] in Theorem 1.4.2, Theorem 1.4.3, Theorem 1.4.4 and Theorem 1.4.5. Theorem 1.4.2 and Theorem 1.4.3 show the relation between the 2-backbone coloring number and the chromatic number in case the backbone is a tree or a path. The 2-backbone coloring number roughly grow like $2k$ and $3k/2$, respectively, where $\chi = k$. Theorem 1.4.4 is a strengthening of Theorem 1.4.2 and Theorem 1.4.3 for the special case of split graphs. Theorem 1.4.5 gives the computational complexity of the 2-backbone coloring number for tree and path backbones.

Theorem 1.4.2. (Broersma, Fomin, Golovach & Woeginger [7])

$$\mathcal{T}_2(k) = 2k - 1 \quad \textit{for } k \geq 1.$$

Theorem 1.4.3. (Broersma, Fomin, Golovach & Woeginger [7])
For $k \geq 1$ the function $\mathcal{P}_2(k)$ takes the following values:

(a) for $1 \leq k \leq 4$: $\mathcal{P}_2(k) = 2k - 1$;

(b) $\mathcal{P}_2(5) = 8$ and $\mathcal{P}_2(6) = 10$;

(c) for $k \geq 7$ and $k = 4t$: $\mathcal{P}_2(4t) = 6t$;

(d) for $k \geq 7$ and $k = 4t+1$: $\mathcal{P}_2(4t+1) = 6t+1$;

(e) for $k \geq 7$ and $k = 4t+2$: $\mathcal{P}_2(4t+2) = 6t+3$;

(f) for $k \geq 7$ and $k = 4t+3$: $\mathcal{P}_2(4t+3) = 6t+5$.

Theorem 1.4.4. (Broersma, Fomin, Golovach & Woeginger [7])
Let $G = (V, E)$ be a split graph with $\chi(G) = k \geq 2$.

(a) For every spanning tree $T = (V, E_T)$ of G,

$$\textsc{bbc}_2(G, T) \leq \begin{cases} 3 & \textit{if } k = 2 \\ k+2 & \textit{if } k \geq 3. \end{cases}$$

(b) For every Hamilton path $P = (V, E_P)$ of G,

$$\textsc{bbc}_2(G, P) \leq \begin{cases} k+1 & \textit{if } k \neq 3 \\ 5 & \textit{if } k = 3. \end{cases}$$

The bounds are tight.

Theorem 1.4.5. (Broersma, Fomin, Golovach & Woeginger [7])

(a) The following problem is polynomially solvable for any $\ell \leq 4$: Given a graph G and a tree backbone T, decide whether $\textsc{bbc}_2(G, T) \leq \ell$.

(b) The following problem is NP-complete for all $\ell \geq 5$: Given a graph G and a path backbone P, decide whether $\textsc{bbc}_2(G, P) \leq \ell$.

Next, we present our results in [46] about the λ-backbone coloring numbers of graphs with star backbones or matching backbones.

Theorem 1.4.6. *For $\lambda \geq 2$ and $k \geq 2$ the function $\mathcal{S}_\lambda(k)$ takes the following values:*

(a) $\mathcal{S}_\lambda(2) = \lambda + 1$;

(b) for $3 \leq k \leq 2\lambda - 3$: $\mathcal{S}_\lambda(k) = \lceil \frac{3k}{2} \rceil + \lambda - 2$;

(c) for $2\lambda - 2 \leq k \leq 2\lambda - 1$ *with* $\lambda \geq 3$: $\mathcal{S}_\lambda(k) = k + 2\lambda - 2$; $\mathcal{S}_2(3) = 5$;

(d) for $k = 2\lambda$ *with* $\lambda \geq 3$: $\mathcal{S}_\lambda(k) = 2k - 1$; $\mathcal{S}_2(4) = 6$;

(e) for $k \geq 2\lambda + 1$: $\mathcal{S}_\lambda(k) = 2k - \lfloor \frac{k}{\lambda} \rfloor$.

Theorem 1.4.7. *For $\lambda \geq 2$ and $k \geq 2$ the function $\mathcal{M}_\lambda(k)$ takes the following values:*

(a) for $2 \leq k \leq \lambda$: $\mathcal{M}_\lambda(k) = \lambda + k - 1$;

(b) for $\lambda + 1 \leq k \leq 2\lambda$: $\mathcal{M}_\lambda(k) = 2k - 2$;

(c) for $k = 2\lambda + 1$: $\mathcal{M}_\lambda(k) = 2k - 3$;

(d) for $k = t(\lambda + 1)$ *with* $t \geq 2$: $\mathcal{M}_\lambda(k) = 2\lambda \cdot t$;

(e) for $k = t(\lambda + 1) + c$ *with* $t \geq 2$, $1 \leq c < \frac{\lambda+3}{2}$: $\mathcal{M}_\lambda(k) = 2\lambda \cdot t + 2c - 1$;

(f) for $k = t(\lambda + 1) + c$ *with* $t \geq 2$, $\frac{\lambda+3}{2} \leq c \leq \lambda$: $\mathcal{M}_\lambda(k) = 2\lambda \cdot t + 2c - 2$.

In [46] we also considered planar graphs. The Four-Color Theorem together with Theorem 1.4.7 implies that $\text{BBC}_2(G, M) \leq 6$ holds for any planar graph G with a perfect matching M. It seems likely that this bound 6 is not best possible, but there are planar graphs showing that we can not improve this bound to 4.

In the last part of [46] we introduced a special kind of 2-backbone coloring and proved Theorem 1.4.8. Let $H = (V, E_H)$ be a backbone of the graph $G = (V, E_G)$. A 2-backbone coloring $f : V \to \{1, \ldots, \ell\}$ of (G, H) is called an *ℓ-cyclic 2-backbone coloring* of (G, H), if there does not exist an edge in E_H that connects two vertices with color 1 and color ℓ in V.

Theorem 1.4.8.

(a) Let G be a planar graph with a matching backbone M. Then (G, M) has a 6-cyclic 2-backbone coloring.

(b) There exist planar graphs that do not have a 5-cyclic 2-backbone coloring where the backbone is a perfect matching.

The three following theorems are our results about the λ-backbone coloring numbers of split graphs with star or matching or tree backbones [8]. Theorem 1.4.11 is a generalization of Theorem 1.4.4(a).

Theorem 1.4.9. *Let $\lambda \geq 2$ and let $G = (V, E)$ be a split graph with $\chi(G) = k \geq 2$. For every star backbone $S = (V, E_S)$ of G,*

$$\text{BBC}_\lambda(G, S) \leq \begin{cases} k + \lambda & \textit{if either } k = 3 \textit{ and } \lambda \geq 2 \textit{ or } k \geq 4 \textit{ and } \lambda = 2 \\ k + \lambda - 1 & \textit{in the other cases.} \end{cases}$$

The bounds are tight.

Theorem 1.4.10. *Let $\lambda \geq 2$ and let $G = (V, E)$ be a split graph with $\chi(G) = k \geq 2$. For every matching backbone $M = (V, E_M)$ of G,*

$$\text{BBC}_\lambda(G, M) \leq \begin{cases} \lambda + 1 & \textit{if } k = 2 \\ k + 1 & \textit{if } k \geq 3 \textit{ and } \lambda \leq \min\{\frac{k}{2}, \frac{k+5}{3}\} \\ k + 2 & \textit{if } k = 9 \textit{ or } k \geq 11 \textit{ and } \frac{k+6}{3} \leq \lambda \leq \lceil \frac{k}{2} \rceil \\ \lceil \frac{k}{2} \rceil + \lambda & \textit{if } k = 3, 5, 7 \textit{ and } \lambda \geq \lceil \frac{k}{2} \rceil \\ \lceil \frac{k}{2} \rceil + \lambda + 1 & \textit{if } k = 4, 6 \textit{ or } k \geq 8 \textit{ and } \lambda \geq \lceil \frac{k}{2} \rceil + 1. \end{cases}$$

The bounds are tight.

Theorem 1.4.11. *Let $\lambda \geq 2$ and let $G = (V, E)$ be a split graph with $\chi(G) = k$. For every tree backbone $T = (V, E_T)$ of G,*

$$\text{BBC}_\lambda(G, T) \leq \begin{cases} 1 & \textit{if } k = 1 \\ 1 + \lambda & \textit{if } k = 2 \\ k + \lambda & \textit{if } k \geq 3. \end{cases}$$

The bounds are tight.

The two following theorems are our results in [46] or [47] about the computational complexity of computing the λ-backbone coloring number of a graph with a star backbone or a matching backbone or a tree backbone or a path backbone. Theorem 1.4.13 is a generalization of Theorem 1.4.5.

Theorem 1.4.12. *Let* $\lambda \geq 2$.

(a) *The following problem is polynomially solvable for any* $\ell \leq \lambda + 1$*: Given a graph* G *and a star backbone* S*, decide whether* $\text{BBC}_\lambda(G, S) \leq \ell$.

(b) *The following problem is NP-complete for all* $\ell \geq \lambda + 2$*: Given a graph* G *and a matching backbone* M*, decide whether* $\text{BBC}_\lambda(G, M) \leq \ell$.

Theorem 1.4.13. *Let* $\lambda \geq 2$.

(a) *The following problem is polynomially solvable for any* $\ell \leq \lambda + 2$*: Given a graph* G *and a spanning tree* T*, decide whether* $\text{BBC}_\lambda(G, T) \leq \ell$.

(b) *The following problem is NP-complete for all* $\ell \geq \lambda + 3$*: Given a graph* G *and a Hamiltonian path* P*, decide whether* $\text{BBC}_\lambda(G, P) \leq \ell$.

Chapter 2

Spanning 2-Connected Subgraphs of Some Classes of Grid Graphs

Abstract In this chapter we define four classes of grid graphs called truncated rectangular grid graphs and 26 classes of grid graphs called alphabet graphs. We determine which of the graphs of the defined classes contain a Hamilton cycle and solve the problem of determining a spanning 2-connected subgraph with as few edges as possible for these graphs.

2.1 Introduction

We recall that the *infinite grid graph* G^∞ is defined by the set of vertices $V = \{(x, y) \mid\ x \in Z,\ y \in Z\ \}$ and the set of edges E between all pairs of vertices from V at Euclidean distance precisely 1. For any integers $s \geq 1$ and $t \geq 1$, the *rectangular grid graph* $R(s, t)$ is the (finite) subgraph of G^∞ induced by $V(s, t) = \{(x, y) \mid 1 \leq x \leq s,\ 1 \leq y \leq t,\ x \in Z,\ y \in Z\}$ (and just containing all edges from G^∞ between pairs of vertices from $V(s, t)$). A *grid graph* is a graph that is isomorphic to a subgraph of $R(s, t)$ induced by a subset of $V(s, t)$ for some integers $s \geq 1$ and $t \geq 1$.

We study the problem of the existence of a Hamilton cycle and the problem of determining a spanning 2-connected subgraph with as few edges as possible for some classes of finite grid graphs with no or a few holes. We define four classes of grid graphs called truncated rectangular grid graphs and 26 classes of grid graphs called alphabet graphs. We give the solution of the second problem for truncated rectangular grid graphs in Section 2.2 and for alphabet graphs in Section 2.3. All solutions are of the same type : first, we use the well-known *Grinberg-condition* and the properties of bipartite graphs to derive a lower bound for the number of edges in a spanning 2-connected subgraph. Secondly, we show by construction that this lower bound is in fact the optimum value.

2.2 Spanning 2-connected subgraphs of truncated rectangular grid graphs

We introduce the classes of grid graphs which we call *truncated rectangular grid graphs*.

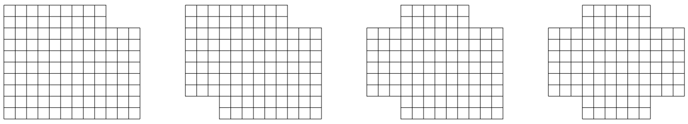
For $s \geq 3$, $t \geq 3$, $0 \leq k \leq min\{s-2, t-2\}$ and $0 \leq l \leq min\{s-2, t-2\}$ we define a 1-*corner truncated rectangular grid graph* $R(s,t)^{-1(k,l)}$ as the subgraph obtained from $R(s,t)$ by deleting $k \times l$ vertices from one corner in $V(s,t)$ together with their incident edges in a natural drawing. For illustration, consider $R(13,11)^{-1(3,2)}$ in Figure 2.1(a).

For $s \geq 6$, $t \geq 6$, $1 \leq k \leq min\{\frac{s-4}{2}, \frac{t-4}{2}\}$ and $1 \leq l \leq min\{\frac{s-4}{2}, \frac{t-4}{2}\}$ we define a 2-*corner truncated rectangular grid graph* $R(s,t)^{-2(k,l)}$ as the subgraph obtained from $R(s,t)$ by deleting $k \times l$ vertices from two opposite corners in $V(s,t)$ together with their incident edges in a natural drawing. For illustration, consider $R(13,11)^{-2(3,2)}$ in Figure 2.1(b).

For $s \geq 6$, $t \geq 6$, $1 \leq k \leq min\{\frac{s-4}{2}, \frac{t-4}{2}\}$ and $1 \leq l \leq min\{\frac{s-4}{2}, \frac{t-4}{2}\}$ we define a 3-*corner truncated rectangular grid graph* $R(s,t)^{-3(k,l)}$ as the subgraph obtained from $R(s,t)$ by deleting $k \times l$ vertices from three corners in $V(s,t)$ together with their incident edges in a natural drawing. For illustration, consider $R(13,11)^{-3(3,2)}$ in Figure 2.1(c).

For $s \geq 6$, $t \geq 6$, $1 \leq k \leq min\{\frac{s-4}{2}, \frac{t-4}{2}\}$ and $1 \leq l \leq min\{\frac{s-4}{2}, \frac{t-4}{2}\}$ we define a 4-*corner truncated rectangular grid graph* $R(s,t)^{-4(k,l)}$ as the subgraph obtained from $R(s,t)$ by deleting $k \times l$ vertices from each corner in

$V(s,t)$ together with their incident edges in a natural drawing. For illustration, consider $R(13,11)^{-4(3,2)}$ in Figure 2.1(d).



(a) (b) (c) (d)

Figure 2.1: Truncated rectangular grid graphs (a) $R(13,11)^{-1(3,2)}$ (b) $R(13,11)^{-2(3,2)}$ (c) $R(13,11)^{-3(3,2)}$ (d) $R(13,11)^{-4(3,2)}$

Spanning 2-connected subgraphs with a minimum number of edges for the 1-corner truncated rectangular grid graph $R(s,t)^{-1(k,k)}$ and for the 4-corner truncated rectangular grid graph $R(s,t)^{-4(k,k)}$ were studied in [38]. Subsequently, in [40] these results were generalized to $R(s,t)^{-1(k,l)}$ and $R(s,t)^{-4(k,l)}$. In [42] we considered the other truncated rectangular grid graphs. We summarize the results in [40] and [42] in the Theorem 2.2.1. It characterizes which of the truncated rectangular grid graphs are hamiltonian and guarantees the existence of a spanning 2-connected subgraph with at most three edges more than their number of vertices.

Theorem 2.2.1. *Let* $R(s,t)^{-1(k,l)}$, $R(s,t)^{-2(k,l)}$, $R(s,t)^{-3(k,l)}$ *and* $R(s,t)^{-4(k,l)}$ *denote the* 1*-corner truncated rectangular grid graph, the* 2*-corner truncated rectangular grid graph, the* 3*-corner truncated rectangular grid graph and the* 4*-corner truncated rectangular grid graph as defined above, respectively. Then:*

(a) $R(s,t)^{-1(k,l)}$ *contains a spanning* 2*-connected subgraph with (at most)* $|V|+1$ *edges and is hamiltonian if and only if both* $s \cdot t$ *and* $k \cdot l$ *are even or both* $s \cdot t$ *and* $k \cdot l$ *are odd.*

(b) $R(s,t)^{-2(k,l)}$ *contains a spanning* 2*-connected subgraph with*
- $|V|$ *edges if* $s \cdot t$ *is even and at least one of* k *and* l *is even if both* s *and* t *are even;*
- $|V|+2$ *edges if* s *and* t *are even and* k *and* l *are odd;*
- $|V|+1$ *edges in all other cases.*

These numbers of edges are all best possible.

(c) $R(s,t)^{-3(k,l)}$ *contains a spanning* 2*-connected subgraph with*
- $|V|$ *edges if both* $s \cdot t$ *and* $k \cdot l$ *are even;*

- $|V| + 2$ *edges if all of* s, t, k *and* l *are odd;*
- $|V| + 1$ *edges in all other cases.*

These numbers of edges are all best possible.

(d) $R(s,t)^{-4(k,l)}$ *contains a spanning* 2*-connected subgraph with (at most)* $|V| + 3$ *edges and is hamiltonian if and only if* $s \cdot t$ *is even. The bound* $|V| + 3$ *is best possible for any odd numbers* s, t, k *and* l.

Proof. We need the following result due to Grinberg for the proof of Theorem 2.2.1.

Lemma 2.2.2. (Grinberg [23])
Suppose a planar graph G *has a Hamilton cycle* H. *Let* G *be drawn in the plane, and let* r_i *denote the number of faces inside* H *bounded by* i *edges in this planar embedding. Let* $r_i^{'}$ *be the number of faces outside* H *bounded by* i *edges. Then the numbers* r_i *and* $r_i^{'}$ *satisfy the following equation.*

$$\sum_i (i-2)(r_i - r_i^{'}) = 0.$$

We use this lemma to show that $R(s,t)^{-1(k,l)}$ and $R(s,t)^{-3(k,l)}$ contain no Hamilton cycle if $s \cdot t$ and $k \cdot l$ have a different parity, and that $R(s,t)^{-2(k,l)}$ and $R(s,t)^{-4(k,l)}$ contain no Hamilton cycle if $s \cdot t$ is odd.

Corollary 2.2.3. $R(m,n)^{-j(k,l)}$ *contains no Hamilton cycle if* ($s \cdot t$ *and* $k \cdot l$ *have a different parity for* $j = 1$ *or* 3) *or* ($s \cdot t$ *is odd for* $j = 2$ *or* 4).

Proof. There is exactly one face with $2(s+t-2)$ edges and there are exactly $(s-1)(t-1) - j \cdot k \cdot l$ faces with four edges in the planar (natural) drawing of the j-corner truncated rectangular grid graph $R(s,t)^{-j(k,l)}$ for $j = 1, 2, 3$ or 4. Let this graph be hamiltonian. Then, by Lemma 2.2.2, we have

$$(2(s+t-2)-2)(-1) + (4-2)(r_4 - r_4^{'}) = 0.$$

Hence

$$r_4 - r_4^{'} = s + t - 3. \tag{2.1}$$

It is easy to check that the number of faces with four edges is

$$r_4 + r_4^{'} = (s-1)(t-1) - j \cdot k \cdot l. \tag{2.2}$$

From equations (2.1) and (2.2) we obtain

$$2r_4 = s \cdot t - j \cdot k \cdot l - 2. \tag{2.3}$$

It implies that either $s \cdot t$ and $k \cdot l$ are even or $s \cdot t$ and $k \cdot l$ are odd for $j = 1$ or 3, and that $s \cdot t$ is even for $j = 2$ or 4. □

Lemma 2.2.4. *$R(s,t)^{-2(k,l)}$ contains no spanning 2-connected subgraph with at most $|V|+1$ edges if both s and t are even and both k and l are odd.*

Proof. First, consider a bipartition (S,T) of $R(s,t)$ for some positive even integers s and t. Assume that one of the corner vertices is in S. Then one easily shows that the opposite corner vertex is also in S, whereas the two other corner vertices are in $|T|$ and that $|S| = |T|$. This can be proved by induction on s and t, and removing the cycle of the outer face if $s, t \geq 4$.

Secondly, consider a bipartition (S,T) of $R(k,l)$ for odd k and l. Assume that one of the corner vertices is in S (if $s, t \geq 3$; otherwise consider an end vertex). Then we can show that all corner vertices (end vertices) are in S, and that $|S| = |T| + 1$. This can be proved by induction on s and t, and removing the cycle of the outer face if $s, t \geq 3$.

So if we remove the two opposite corner $R(k,l)$'s from $R(s,t)$, we reduce $|S|$ by two more units than $|T|$, implying that $R(s,t)^{-2(k,l)}$ has a bipartition (S',T') with $|T'| = |S'| + 2$. In any spanning 2-connected subgraph G of $R(s,t)^{-2(k,l)}$ all vertices in T' have degree at least 2, hence $|E(G)| \geq 2|T'| = |T'| + |S'| + 2 = |V(G)| + 2$. This completes the proof of Lemma 2.2.4. □

Lemma 2.2.5. *$R(s,t)^{-3(k,l)}$ contains no spanning 2-connected subgraph with at most $|V|+1$ edges if all of s, t, k and l are odd.*

Proof. Consider a bipartition (S,T) of $R(s,t)$ for odd s and t. Assume that one of the corner vertices is in S. By the same arguments as in the proof of Lemma 2.2.4, then all corner vertices are in S, and $|S| = |T| + 1$. The same holds for $R(k,l)$ if k and l are odd. So if we remove the three corner $R(k,l)$'s from $R(s,t)$, we reduce $|S|$ by three more units than $|T|$, implying that $R(s,t)^{-3(k,l)}$ has a bipartition (S',T') with $|T'| = |S'|+2$. In any spanning 2-connected subgraph G of $R(s,t)^{-3(k,l)}$ all vertices in T' have degree at least 2, hence $|E(G)| \geq 2|T'| = |T'| + |S'| + 2 = |V(G)| + 2$. This completes the proof of Lemma 2.2.5. □

Lemma 2.2.6. *$R(s,t)^{-4(k,l)}$ contains no spanning 2-connected subgraph with at most $|V|+2$ edges if s,t,k and l are odd.*

Proof. First, consider a bipartition (S,T) of $R(s,t)$ for odd s and t. Assume that one of the corner vertices is in S. By the same arguments as in the proof of Lemma 2.2.4, then all corner vertices are in S, and $|S| = |T| + 1$. The same holds for $R(k,l)$ if k and l are odd. So if we remove the four corner $R(k,l)$'s from $R(s,t)$, we reduce $|S|$ by four more units than $|T|$, implying that $R(s,t)^{-4(k,l)}$ has a bipartition (S',T') with $|T'| = |S'| + 3$. In any spanning 2-connected subgraph G for $R(s,t)^{-4(k,l)}$ all vertices in T' have degree at least 2, hence $|E(G)| \geq 2|T'| = |T'| + |S'| + 3 = |V(G)| + 3$. This completes the proof of Lemma 2.2.6. □

We complete the proof of Theorem 2.2.1 by showing, through construction, the existence of a Hamilton cycle or a spanning 2-connected subgraph with at most $|V|+3$ edges, in all cases where $s = 12$ or 13, $t = 10$ or 11, $k = 2$ or 3 and $l = 1, 2$ or 3. Meanwhile, for other values of s, t, k and l, it is not difficult to see, from the patterns in the figures that now follow, how to extend the solutions.

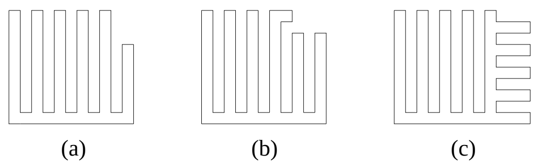
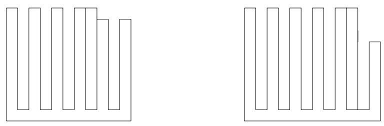


Figure 2.2: Hamilton cycles for (a) $R(12,11)^{-1(2,3)}$ (b) $R(12,11)^{-1(3,2)}$ (c) $R(13,11)^{-1(3,1)}$

A Hamilton cycle for $R(12,11)^{-1(2,3)}$ is shown in Figure 2.2(a). The pattern in this figure can be used for finding a Hamilton cycle for the 1-corner truncated rectangular grid graph for either (any numbers t and l, and any even numbers s and k) or (any numbers s and k, and any even numbers t and l). In Figure 2.2(b) we show a Hamilton cycle for $R(12,11)^{-1(3,2)}$. The pattern in Figure 2.2(b) can be used for finding a Hamilton cycle for the 1-corner truncated rectangular grid graph for either (any number t, any even numbers s and l, and any odd number k) or (any number s, any even numbers t and k, and any odd number l). Meanwhile, in Figure 2.2(c) we show a Hamilton cycle for $R(13,11)^{-1(3,1)}$. The pattern in Figure 2.2(c) can be used for finding a

Hamilton cycle for the 1-corner truncated rectangular grid graph for any odd numbers s, t, k and l.



(a) (b)

Figure 2.3: Spanning 2-connected subgraphs with $|V| + 1$ edges for (a) $R(12, 11)^{-1(3,1)}$ (b) $R(13, 11)^{-1(2,3)}$

A spanning 2-connected subgraph for $R(12, 11)^{-1(3,1)}$ with $|V| + 1$ edges is shown in Figure 2.3(a). The pattern in this figure can be used for finding such a spanning subgraph with $|V| + 1$ edges for the 1-corner truncated rectangular grid graph for any even number s or t and for any odd numbers k and l. In Figure 2.3(b) we show a spanning 2-connected subgraph with $|V| + 1$ edges for $R(13, 11)^{-1(2,3)}$. The pattern in Figure 3(b) can be used for finding a spanning 2-connected subgraph with $|V| + 1$ edges for the 1-corner truncated rectangular grid graph for any odd numbers s and t and for any even number k or l.

A Hamilton cycle for $R(12, 11)^{-2(2,3)}$ is shown in Figure 2.4(a). The pattern in this figure can be used for finding a Hamilton cycle for the 2-corner truncated rectangular grid graph for either (any numbers t and l, and any even numbers s and k) or (any numbers s and k, and any even numbers t and l). In Figure 2.4(b) we show a Hamilton cycle for $R(12, 11)^{-2(3,2)}$. The pattern in Figure 2.4(b) can be used for finding a Hamilton cycle for the 2-corner truncated rectangular grid graph for either (any even numbers s and l, and any odd numbers t and k) or (any even numbers t and k, and any odd numbers s and l). In Figure 2.4(c) we show a Hamilton cycle for $R(12, 11)^{-2(3,3)}$. The pattern in Figure 2.4(c) can be used for finding a Hamilton cycle for the 2-corner truncated rectangular grid graph for either (any even number s, and any odd numbers t, k and l) or (any even number t, and any odd numbers s, k and l).

A spanning 2-connected subgraph for $R(13, 11)^{-2(3,2)}$ with $|V| + 1$ edges is shown in Figure 2.5(a). The pattern in this figure can be used for finding such a spanning subgraph with $|V| + 1$ edges for the 2-corner truncated rectangular grid graph for either (any number l, and any odd numbers s, t and k) or (any number k, and any odd numbers s, t and l). In Figure 2.5(b) we show a spanning 2-connected subgraph with $|V| + 1$ edges for $R(13, 11)^{-2(2,2)}$. The pattern

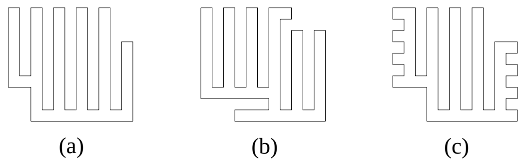


Figure 2.4: Hamilton cycles for (a) $R(12,11)^{-2(2,3)}$ (b) $R(12,11)^{-2(3,2)}$ (c) $R(12,11)^{-2(3,3)}$

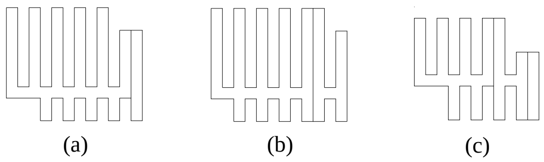


Figure 2.5: Spanning 2-connected subgraphs for (a) $R(13,11)^{-2(3,2)}$ with $|V|+1$ edges (b) $R(13,11)^{-2(2,2)}$ with $|V|+1$ edges (c) $R(12,10)^{-2(3,3)}$ with $|V|+2$ edges

in Figure 2.5(b) can be used for finding a spanning 2-connected subgraph with $|V|+1$ edges for the 2-corner truncated rectangular grid graph for any even numbers k and l and any odd numbers s and t. In Figure 2.5(c) we show a spanning 2-connected subgraph with $|V|+2$ edges for $R(12,10)^{-2(3,3)}$. The pattern in this figure can be used for finding a spanning 2-connected subgraph with $|V|+2$ edges for the 2-corner truncated rectangular grid graph for any even numbers s, t, and any odd numbers k and l. This is the optimum value for the minimum number of edges in such a spanning 2-connected subgraph.

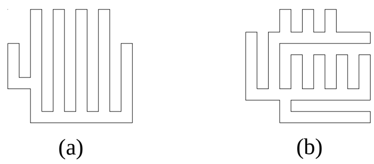
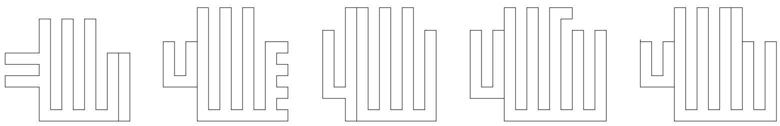


Figure 2.6: Hamilton cycles for (a) $R(12,11)^{-3(2,3)}$ (b) $R(12,11)^{-3(3,2)}$

Hamilton cycles for $R(12,11)^{-3(2,3)}$ and $R(12,11)^{-3(3,2)}$ are shown in Figure 2.6. The pattern in Figure 2.6(a) can be used for finding a Hamilton cycle for the 3-corner truncated rectangular grid graph for either (any numbers t and l, and any even numbers s and k) or (any numbers s and k, and any even numbers t and l). The pattern in Figure 2.6(b) can be used for finding a Hamilton cycle for the 3-corner truncated rectangular grid graph for either (any even numbers s and l, and any odd numbers t and k) or (any even numbers t and k, and any odd numbers s and l).



(a) (b) (c) (d) (e)

Figure 2.7: Spanning 2-connected subgraphs for (a) $R(12,10)^{-3(3,3)}$ with $|V|+1$ edges (b) $R(12,11)^{-3(3,3)}$ with $|V|+1$ edges (c) $R(13,11)^{-3(2,2)}$ with $|V|+1$ edges (d) $R(13,11)^{-3(3,2)}$ with $|V|+1$ edges (e) $R(13,11)^{-3(3,3)}$ with $|V|+2$ edges

In Figure 2.7(a), Figure 2.7(b), Figure 2.7(c) and Figure 2.7(d) we show spanning 2-connected subgraphs for the 3-corner truncated rectangular grid graphs $R(12,10)^{-3(3,3)}$, $R(12,11)^{-3(3,3)}$, $R(13,11)^{-3(2,2)}$ and $R(13,11)^{-3(3,2)}$, respectively, with $|V|+1$ edges. The pattern in Figure 2.7(a) can be used for finding such a spanning 2-connected subgraph for any even numbers s and t, and any odd numbers k and l. The pattern in Figure 2.7(b) can be used for finding such a spanning 2-connected subgraph for either (any even number s, and any odd numbers t, k and l) or (any even number t, and any odd numbers s, k and l). The pattern in Figure 2.7(c) can be used for finding such a spanning 2-connected subgraph for any even numbers k and l, and any odd numbers s and t. The pattern in Figure 2.7(d) can be used for finding such a spanning 2-connected subgraph for either (any even number l, and any odd numbers s, t and k) or (any even number k, and any odd numbers s, t and l). In Figure 2.7(e) we show a spanning 2-connected subgraph with $|V|+2$ edges for $R(13,11)^{-3(3,3)}$. The pattern in this figure can be used for finding a spanning 2-connected subgraph with $|V|+2$ edges for the 3-corner truncated rectangular grid graph for any odd numbers s, t, k and l. This is the optimum value for the minimum number of edges in such a spanning 2-connected subgraph.

Hamilton cycles for $R(12,11)^{-4(2,3)}$ and $R(12,11)^{-4(3,2)}$ are shown in Figure 2.8. The pattern in Figure 2.8(a) can be used for finding a Hamilton cycle for the 4-corner truncated rectangular grid graph for either (any numbers t and l, and any even numbers s and k) or (any numbers s and k, and any even numbers t and l). Meanwhile, the pattern in Figure 2.8(b) can be used for finding a Hamilton cycle for the 4-corner truncated rectangular grid graph for

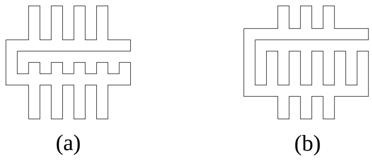


Figure 2.8: Hamilton cycles for (a) $R(12,11)^{-4(2,3)}$ (b) $R(12,11)^{-4(3,2)}$

either (any numbers t and l, any even number s, and any odd number k) or (any numbers s and k, any even number t, and any odd number l).

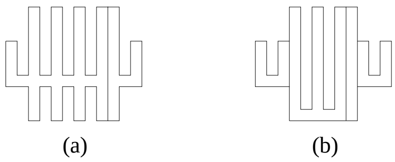


Figure 2.9: Spanning 2-connected subgraphs for (a) $R(13,11)^{-4(2,3)}$ with $|V|+1$ edges (b) $R(13,11)^{-4(3,3)}$ with $|V|+3$ edges

In Figure 2.9(a), we show a spanning 2-connected subgraph with $|V|+1$ edges for $R(13,11)^{-4(2,3)}$. The pattern in this figure can be used for finding a spanning 2-connected subgraph with $|V|+1$ edges for the 4-corner truncated rectangular grid graph for any odd numbers s and t, and for any even number k or l. In Figure 2.9(b) we show a spanning 2-connected subgraph with $|V|+3$ edges for $R(13,11)^{-4(3,1)}$. The pattern in this figure can be used for finding a spanning 2-connected subgraph with $|V|+3$ edges for the 4-corner truncated rectangular grid graph for any odd numbers s, t, k and l. This is the optimum value for the minimum number of edges in such a spanning 2-connected subgraph. □

2.3 Spanning 2-connected subgraphs of alphabet graphs

We now introduce the 26 classes of grid graphs which we call *alphabet graphs*. For every letter λ of the alphabet $\{a, b, \ldots, z\}$ we define a corresponding subgraph $\Lambda_{m,n}$ of $R(3m-2, 5n-4)$ for all $m \geq 3$, $n \geq 3$. These alphabet graphs $\{A_{m,n}, B_{m,n}, \ldots, Z_{m,n}\}$ are shown in Figure 2.10 for $m = 4$ and $n = 3$. It is clear from these figures how these graphs should be extended for other values of m and n. We avoid the tedious details of defining all these 26 graph classes formally. Note that the extension of these classes to $m = 2$ or $n = 2$ causes problems with the definition of grid graphs: for instance, the natural definition of $E_{2,2}$ would not result in an induced subgraph of G^{∞}.

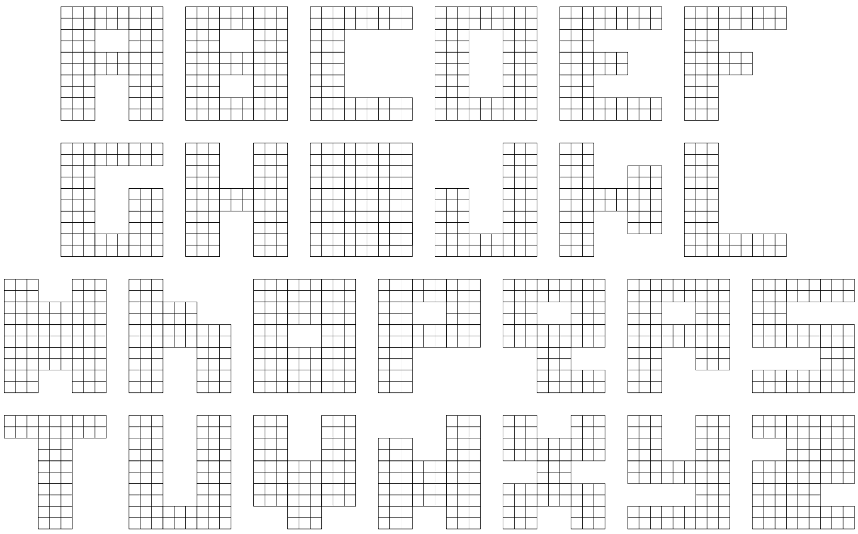


Figure 2.10: Alphabet graphs in order from A to Z for $m = 4$ and $n = 3$

Notice that from these 26 classes, there is one class of alphabet graphs with two holes, namely the graph $B_{m,n}$; six classes with one hole, namely the graphs $A_{m,n}, D_{m,n}, O_{m,n}$, $P_{m,n}, Q_{m,n}$ and $R_{m,n}$; the remaining 19 classes contain no holes, i.e. are solid grid graphs.

We refer to these classes in the next result just by the capital letters, omitting the indices. Spanning 2-connected subgraphs with a minimum number of edges for the alphabet graphs for $m = n$ were studied in [39]. It is a continuation of the work started in [37]. Subsequently, in [42] these results were generalized to the following theorem.

Theorem 2.3.1. *Let $m \geq 3$ and $n \geq 3$. Let $A, B, \ldots, Z$ denote the alphabet graphs $A_{m,n}, B_{m,n}, \ldots, Z_{m,n}$ as defined above. Then:*

(a) A, D, O and P are hamiltonian.

(b) *E and F contain a spanning* 2*-connected subgraph with (at most)* $|V|+1$ *edges and are hamiltonian if and only if n is even.*

(c) *N contains a spanning* 2*-connected subgraph with (at most)* $|V|+1$ *edges and is hamiltonian if and only if m and n have a different parity.*

(d) *Q contains a spanning* 2*-connected subgraph with (at most)* $|V|+1$ *edges and is hamiltonian if and only if m is odd or n is even.*

(e) *R contains a spanning* 2*-connected subgraph with (at most)* $|V|+1$ *edges and is hamiltonian if and only if m is even or n is odd.*

(f) *W contains a spanning* 2*-connected subgraph with*
- $|V|$ *edges if m is even;*
- $|V|+1$ *edges if both m and n are odd;*
- $|V|+2$ *edges if m is odd and n is even.*

These numbers of edges are all best possible.

(g) *X contains a spanning* 2*-connected subgraph with*
- $|V|$ *edges if either (m is even) or (m is odd,* $m \geq 7$ *and n is even);*
- $|V|+1$ *edges if either (m and n are odd) or (m=5 and n is even);*
- $|V|+2$ *edges if m=3 and n is even.*

(h) *The remaining alphabet graphs contain a spanning* 2*-connected subgraph with (at most)* $|V|+1$ *edges and are hamiltonian if and only if* $m \cdot n$ *is even.*

Proof. First, we prove the following corollaries of Lemma 2.2.2. After that we prove Lemma 2.3.7. Finally, we show, through construction, spanning 2-connected subgraphs with as few edges as possible for all alphabet graphs.

Corollary 2.3.2. *E and F contain no Hamilton cycle if n is odd.*

Proof. We divide the proof into two cases.

Case 1 We consider the alphabet graph E. There is exactly one face with $12(m-1)+10(n-1)$ edges and there are exactly $10(m-1)(n-1)$ faces with four edges in the planar (natural) drawing of this graph. Let E be hamiltonian. Then, by Lemma 2.2.2, we have

$$(12(m-1)+10(n-1)-2)(-1)+(4-2)(r_4-r_4^{'})=0.$$

Hence

$$r_4 - r_4^{'} = 6m + 5n - 12. \tag{2.4}$$

It is known that the number of faces with four edges is

$$r_4 + r_4^{'} = 10m \cdot n - 10m - 10n + 10. \tag{2.5}$$

From equations (2.4) and (2.5) we obtain

$$2r_4 = 10m \cdot n - 4m - 5n - 2. \tag{2.6}$$

So, n is even.

Case 2 We consider the alphabet graph F. There is exactly one face with $8(m-1)+10(n-1)$ edges and there are exactly $8(m-1)(n-1)$ faces with four edges in the planar (natural) drawing of this graph. Let F be hamiltonian. Then, by Lemma 2.2.2 and using a method similar to that used in the proof of Case 1, we obtain

$$2r_4 = 8m \cdot n - 4m - 3n - 2. \tag{2.7}$$

So, n is even. □

Corollary 2.3.3. *N contains no Hamilton cycle if m and n have the same parity.*

Proof. There is exactly one face with $6(m-1)+14(n-1)$ edges and there are exactly $10(m-1)(n-1)$ faces with four edges in the planar (natural) drawing of the graph N. Let N be hamiltonian. Then, by Lemma 2.2.2 and using a method similar to that used in the proof of Corollary 2.3.2, we obtain

$$2r_4 = 10m \cdot n - 7m - 3n - 1. \tag{2.8}$$

So, m and n have a different parity. □

Corollary 2.3.4. *Q contains no Hamilton cycle if m is even and n is odd.*

Proof. It is easy to check that there is exactly one face with $8(m-1)+10(n-1)$ edges, one face with $2(m+n-2)$ edges and there are exactly $11(m-1)(n-1)$ faces with four edges in the planar (natural) drawing of the graph Q. Let Q be hamiltonian. Then, by Lemma 2.2.2 and using a method similar to that used in the proof of Corollary 2.3.2, we obtain

$$2r_4 = 11m \cdot n - 7m - 6n + 1 - (m+n-3)(r_{2(m+n-2)} - r'_{2(m+n-2)}). \quad (2.9)$$

So, m is odd or n is even since $(r_{2(m+n-2)} - r'_{2(m+n-2)})$ is -1 or 1. □

Corollary 2.3.5. *R contains no Hamilton cycle if m is odd and n is even.*

Proof. We can check that there is exactly one face with $6(m-1)+12(n-1)$ edges, one face with $2(m+n-2)$ edges and there are exactly $11(m-1)(n-1)$ faces with four edges in the planar (natural) drawing of the graph R. Let R be hamiltonian. Then, by Lemma 2.2.2 and using a method similar to that used in the proof of Corollary 2.3.2, we obtain

$$2r_4 = 11m \cdot n - 8m - 5n + 1 - (m+n-3)(r_{2(m+n-2)} - r'_{2(m+n-2)}). \quad (2.10)$$

So, m is even or n is odd since $(r_{2(m+n-2)} - r'_{2(m+n-2)})$ is -1 or 1. □

Corollary 2.3.6. *B, C, G, H, I, J, K, L, M, S, T, U, V, W, X, Y and Z contain no Hamilton cycle if* $m \cdot n$ *is odd.*

Proof. We divide the proof into two cases.

Case 1 We consider the alphabet graph B. There is exactly one face with $6(m-1)+10(n-1)$ edges, there are two faces with $2(m+n-2)$ edges and there are exactly $13(m-1)(n-1)$ faces with four edges in the planar (natural) drawing of B. Let B be hamiltonian. Then, by Lemma 2.2.2 and using a method similar to that used in the proof of Corollary 2.3.2, we obtain

$$2r_4 = 13m \cdot n - 10m - 8n + 4 - (m+n-3)(r_{2(m+n-2)} - r'_{2(m+n-2)}). \quad (2.11)$$

So, $m \cdot n$ is even since $(r_{2(m+n-2)} - r'_{2(m+n-2)})$ is $-2, 0$ or 2.

Case 2 We consider the alphabet graphs C, G, H, I, J, K, L, M, S, T, U, V, W, X, Y and Z. They are solid grid graphs. So the only faces to be considered in the planar (natural) drawing of every one of these graphs are the single outer face and the faces with four edges. Let these graphs be hamiltonian. The number of edges in the outer face is always even since they form a cycle and the graphs are bipartite. This number is then always $2x(m-1)+2y(n-1)$ for some positive integers x and y. The number of faces with four edges is always of the form $z(m-1)(n-1)$ for some positive integer z. Then, by Lemma 2.2.2 and using a method similar to that used in the proof of Corollary 2.3.2, we obtain

$$2r_4 = z \cdot m \cdot n + (x-z)m + (y-z)n + (z-x-y-1). \tag{2.12}$$

Since

$$(x,y,z) = \begin{cases} (5,5,9) & \text{for } C \\ (5,7,11) & \text{for } G, X, Y \\ (3,9,11) & \text{for } H, U \\ (3,5,15) & \text{for } I \\ (3,7,9) & \text{for } J, K \\ (3,5,7) & \text{for } L, T \\ (3,7,13) & \text{for } M \\ (7,5,11) & \text{for } S \\ (3,7,11) & \text{for } V, W \\ (5,5,13) & \text{for } Z, \end{cases}$$

z is odd, and $x-z$, $y-z$ and $z-x-y-1$ are even. So, $m \cdot n$ is even. □

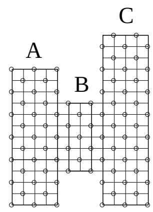


Figure 2.11: Partition of the alphabet graph $W_{5,4}$

Lemma 2.3.7. *W contains no spanning 2-connected subgraph with at most $|V|+1$ edges if m is odd and n is even.*

Proof. Consider the alphabet graph W for odd m and even n. We can partition this graph into three rectangles; name them A (on the left), B (in the middle) and C (on the right). A is $R(m, 4n-3)$, B is $R(m-2, 2n-1)$ and C is $R(m, 5n-4)$. For illustration, look at the partition of the alphabet graph $W_{5,4}$ in Figure 2.11. All of the rectangles are bipartite graphs with a bipartition of the vertices, say in S and T, where we start with S in a corner vertex of A. It is easy to check that A and B have one more vertex from S than from T, whereas C has the same number of vertices from S and from T. So, $|V(W) \cap S| = |V(W) \cap T| + 2$. In any spanning 2-connected subgraph G of W all vertices in S have degree at least 2, hence $|E(G)| \geq 2|S| = |S| + |T| + 2 = |V(G)| + 2$. This completes the proof of Lemma 2.3.7. □

We complete the proof of Theorem 2.3.1 by showing, through construction, the existence of a Hamilton cycle or a spanning 2-connected subgraph with at most $|V| + 2$ edges, in all cases where $m = 3, 5, 6$ or 7 and $n = 4$ or 5. Meanwhile, for other values of m and n, it is not difficult to see, from the patterns in the figures that now follow, how to extend the solutions.

Hamilton cycles for the alphabet graphs A, D, O, and P are shown in Figure 2.12 for $m = 5$ and $n = 4$, in Figure 2.13 for $m = 6$ and $n = 4$, and in Figure 2.14 for $m = 7$ and $n = 5$. The patterns in Figure 2.12 can be used for finding Hamilton cycles for these graphs for any odd number m and any even number n; the patterns in Figure 2.13 can be used for finding Hamilton cycles for these graphs for any even number m and any number n; and the patterns in Figure 2.14 can be used for finding Hamilton cycles for these graphs for any odd numbers m and n.

In Figure 2.15, we show Hamilton cycles for the alphabet graphs $E_{5,4}$, $F_{5,4}$, $N_{5,4}$, $N_{6,5}$, $Q_{6,4}$, $Q_{7,5}$, $R_{6,4}$ and $R_{7,5}$. The patterns in Figure 2.15(a) and Figure 2.15(b) can be used for finding Hamilton cycles for the alphabet graphs E and F, respectively, for any number m and any even number n. The patterns in Figure 2.15(c) and Figure 2.15(d) can be used for finding Hamilton cycles for

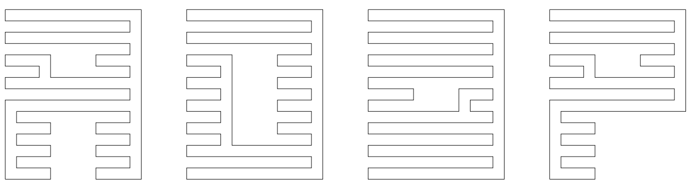


Figure 2.12: Hamilton cycles for the alphabet graphs A, D, O, and P for $m = 5$ and $n = 4$

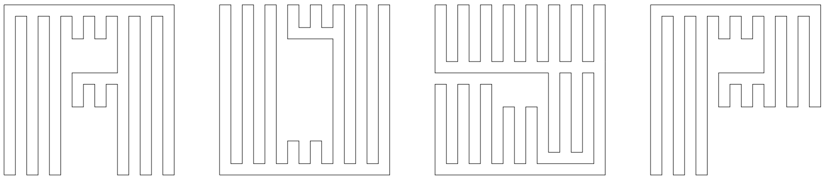


Figure 2.13: Hamilton cycles for the alphabet graphs A, D, O, and P for $m = 6$ and $n = 4$

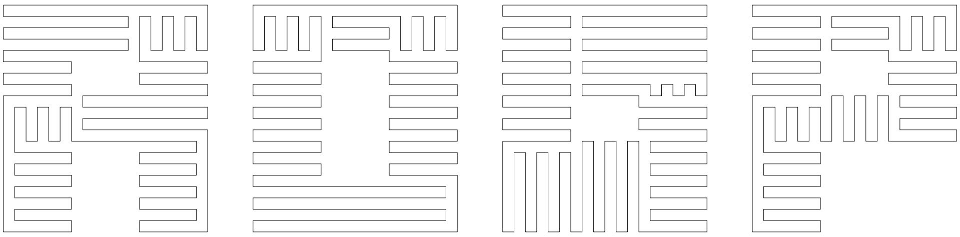


Figure 2.14: Hamilton cycles for the alphabet graphs A, D, O, and P for $m = 7$ and $n = 5$

the alphabet graph N (the pattern in Figure 2.15(c) for any odd number m and any even number n, the pattern in Figure 2.15(d) for any even number m and any odd number n). The patterns in Figure 2.15(e) and Figure 2.15(f) can be used for finding Hamilton cycles for the alphabet graph Q (the pattern in Figure 2.15(e) for any number m and any even number n, the pattern in Figure 2.15(f) for any odd numbers m and n). The patterns in Figure 2.15(g) and Figure 2.15(h) can be used for finding Hamilton cycles for the alphabet graph R (the pattern in Figure 2.15(g) for any even number m and any number n, the pattern in Figure 2.15(h) for any odd numbers m and n).

In Figure 2.16, we show spanning 2-connected subgraphs with $|V| + 1$ edges for the alphabet graphs $E_{6,5}$, $F_{6,5}$, $N_{6,4}$, $N_{7,5}$, $Q_{6,5}$ and $R_{5,4}$. The pattern in Figure 2.16(a) can be used for determining such a spanning subgraph for the alphabet graph E for any number m and any odd number n. The pattern in Figure 2.16(b) can be used for determining such a spanning subgraph for the alphabet graph F for any number m and any odd number n. The patterns in Figure 2.16(c) and Figure 2.16(d) can be used for finding such a spanning subgraph for the alphabet graph N (the pattern in Figure 2.16(c) for any even numbers m and n, the pattern in Figure 2.16(d) for any odd numbers m and n). The pattern in Figure 2.16(e) can be used for determining such

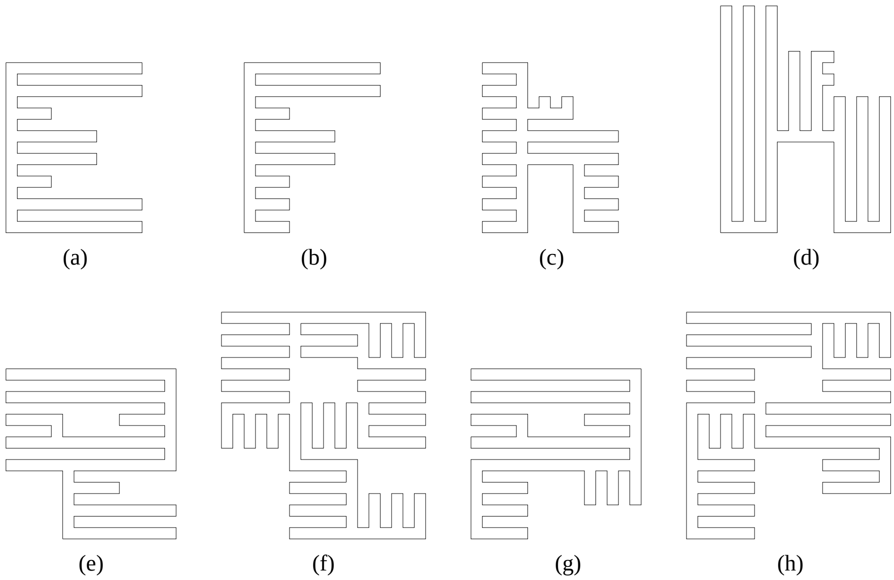


Figure 2.15: Hamilton cycles for the alphabet graphs (a) $E_{5,4}$ (b) $F_{5,4}$ (c) $N_{5,4}$ (d) $N_{6,5}$ (e) $Q_{6,4}$ (f) $Q_{7,5}$ (g) $R_{6,4}$ (h) $R_{7,5}$

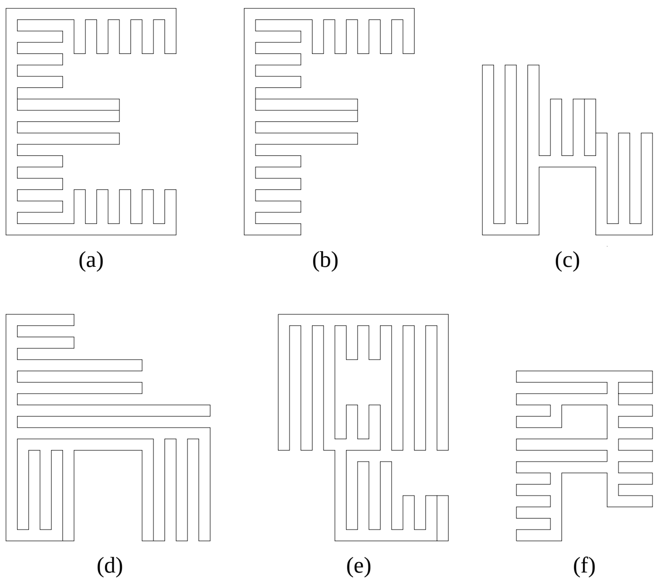


Figure 2.16: Spanning 2-connected subgraphs with $|V| + 1$ edges for the alphabet graphs (a) $E_{6,5}$ (b) $F_{6,5}$ (c) $N_{6,4}$ (d) $N_{7,5}$ (e) $Q_{6,5}$ (f) $R_{5,4}$

a spanning subgraph for the alphabet graph Q for any even number m and any odd number n. The pattern in Figure 2.16(f) can be used for determining such a spanning subgraph for the alphabet graph R for any odd number m and any even number n.

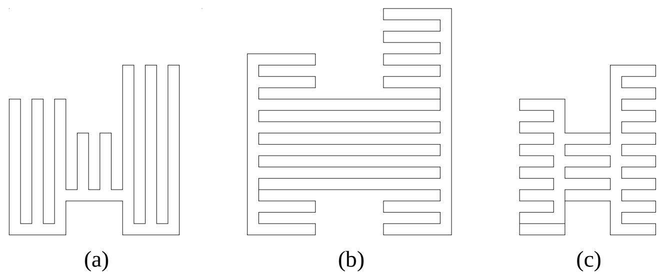


Figure 2.17: (a) A Hamilton cycle for the alphabet graph $W_{6,4}$ (b) A spanning 2-connected subgraph with $|V| + 1$ edges for the alphabet graph $W_{7,5}$ (c) A spanning 2-connected subgraph with $|V| + 2$ edges for the alphabet graph $W_{5,4}$

We show a Hamilton cycle for the alphabet graph $W_{6,4}$ in Figure 2.17(a). The pattern in Figure 2.17(a) can be used for finding a Hamilton cycle for the alphabet graph W for any even number m and any number n. In Figure 2.17(b) is shown a spanning 2-connected subgraph with $|V|+1$ edges for the alphabet graph $W_{7,5}$. The pattern in Figure 2.17(b) can be used for determining such a spanning subgraph for the alphabet graph W for any odd numbers m and n. In Figure 2.17(c) is shown a spanning 2-connected subgraph with $|V| + 2$ edges for the alphabet graph $W_{5,4}$. The pattern in Figure 2.17(c) can be used for determining a spanning 2-connected subgraph with $|V| + 2$ edges for the alphabet graph W for any odd number m and any even number n. This is the optimum value for the minimum number of edges in such a spanning 2-connected subgraph.

We show Hamilton cycles for the alphabet graphs $X_{6,4}$ in Figure 2.18(a) and $X_{7,4}$ in Figure 2.18(b). The pattern in Figure 2.18(a) can be used for finding a Hamilton cycle for the alphabet graph X for any even number m and any number n, whereas the pattern in Figure 2.18(b) can be used for finding a Hamilton cycle for any odd number m, $m \geq 7$ and any even number n. Meanwhile, in Figure 2.18(c) and Figure 2.18(d) are shown spanning 2-connected subgraphs with $|V| + 1$ edges for the alphabet graphs $X_{7,5}$ and $X_{5,4}$, respectively. The patterns in Figure 2.18(c) and Figure 2.18(d) can be used for determining

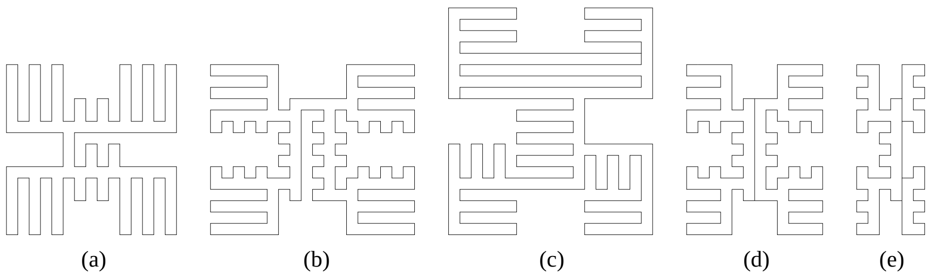


Figure 2.18: Hamilton cycles for the alphabet graphs (a) $X_{6,4}$ (b) $X_{7,4}$; Spanning 2-connected subgraphs with $|V|+1$ edges for the alphabet graphs c) $X_{7,5}$ (d) $X_{5,4}$; (e) A spanning 2-connected subgraph with $|V|+2$ edges for the alphabet graph $X_{3,4}$

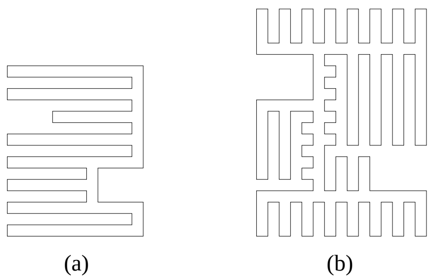


Figure 2.19: Hamilton cycles for the alphabet graphs (a) $Z_{5,4}$ (b) $Z_{6,5}$

spanning 2-connected subgraphs with $|V|+1$ edges for the alphabet graph X (the pattern in Figure 2.18(c) for any odd numbers m and n, the pattern in Figure 2.18(d) for $m=5$ and any even number n). In Figure 2.18(e) is shown a spanning 2-connected subgraph with $|V|+2$ edges for the alphabet graph $X_{3,4}$. The pattern in Figure 2.18(e) can be used for determining a spanning 2-connected subgraph with $|V|+2$ edges for the alphabet graph X for $m=3$ and any even number n. We are not sure that this is the optimum value for the minimum number of edges in a spanning 2-connected subgraph.

We show Hamilton cycles for the alphabet graph $Z_{5,4}$ in Figure 2.19(a) and $Z_{6,5}$ in Figure 2.19(b). The pattern in Figure 2.19(a) can be used for finding a Hamilton cycle for the alphabet graph Z for any number m and any even number n, whereas the pattern in Figure 2.19(b) can be used for finding a Hamilton cycle for any even number m and any odd number n.

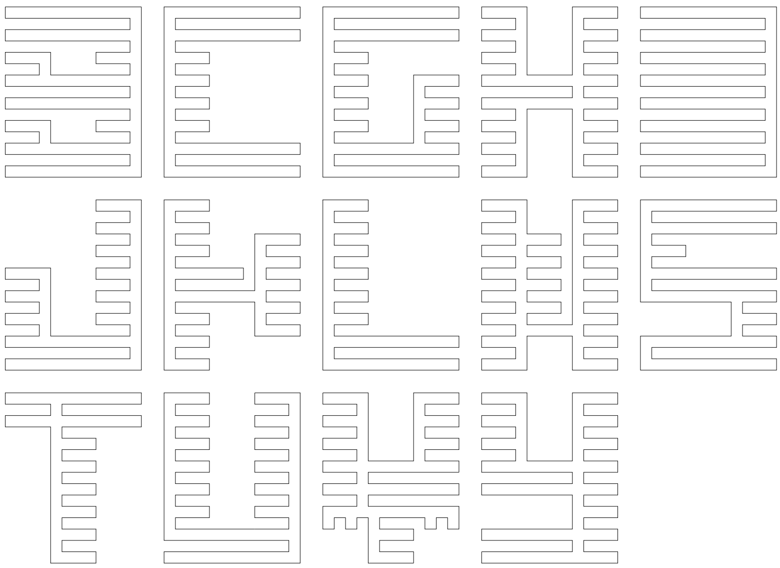


Figure 2.20: Hamilton cycles for the alphabet graphs B, C, G, H, I, J, K, L, M, S, T, U, V and Y for $m = 5$ and $n = 4$

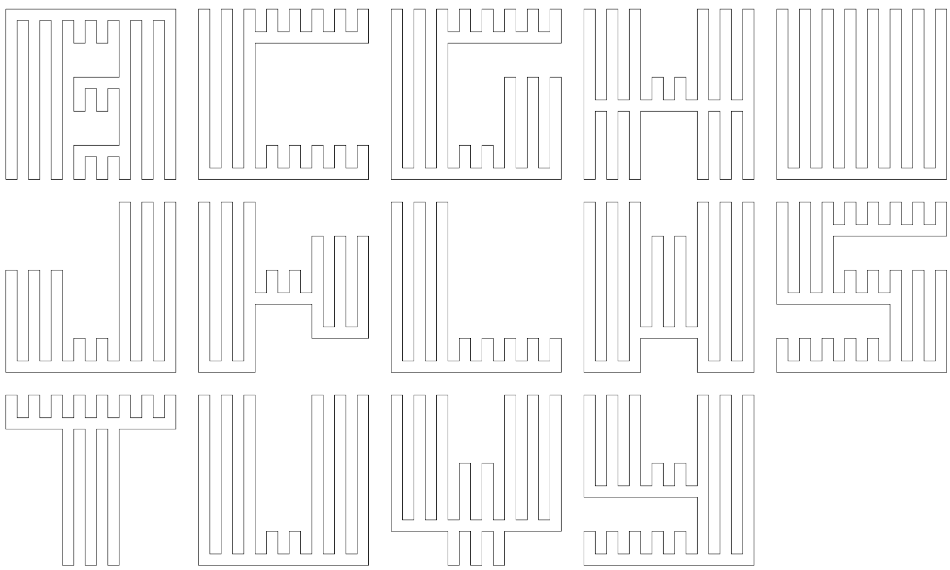


Figure 2.21: Hamilton cycles for the alphabet graphs B, C, G, H, I, J, K, L, M, S, T, U, V and Y for $m = 6$ and $n = 4$

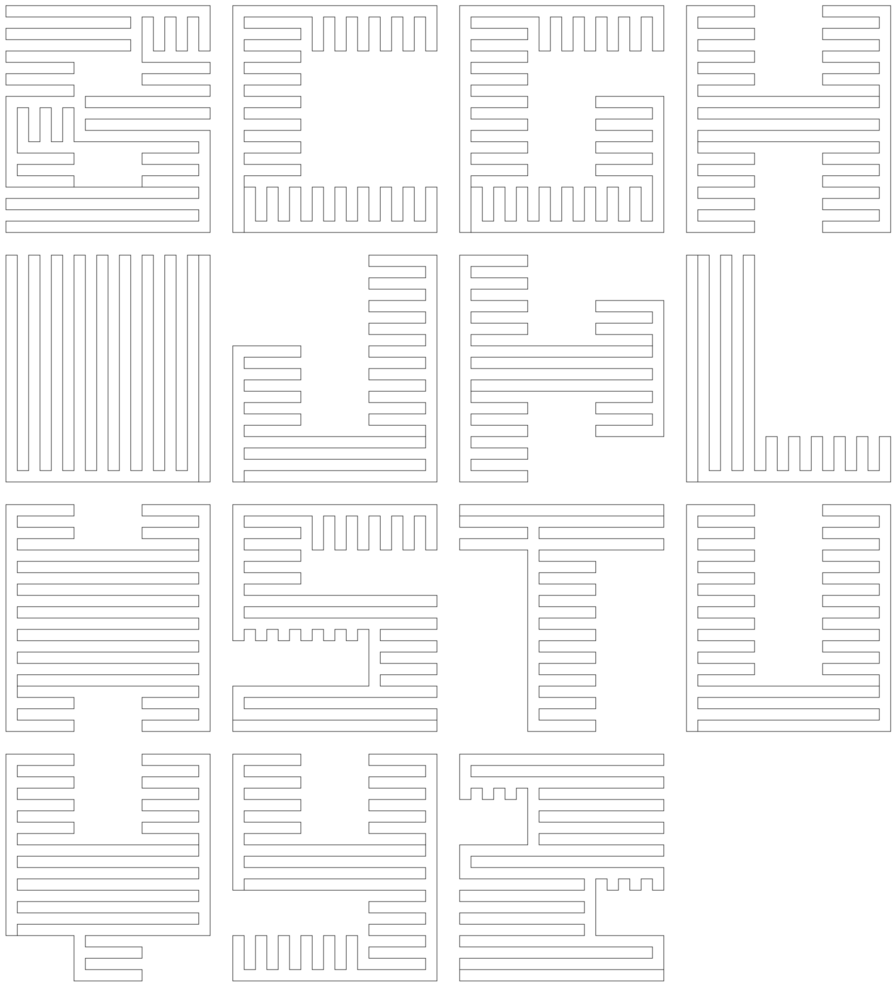


Figure 2.22: Spanning 2-connected subgraphs with $|V|+1$ edges for the alphabet graphs B, C, G, H, I, J, K, L, M, S, T, U, V, Y and Z for $m = 7$ and $n = 5$

Hamilton cycles for the remaining alphabet graphs are shown in Figure 2.20 for $m = 5$ and $n = 4$ and in Figure 2.21 for $m = 6$ and $n = 4$. The patterns in Figure 2.20 can be used for finding Hamilton cycles for these graphs for any odd number m and any even number n. The patterns in Figure 2.21 can be used for determining Hamilton cycles for these graphs for any even number m and any number n. Finally, in Figure 2.22 we show spanning 2-connected subgraphs with $|V| + 1$ edges for the alphabet graphs in (viii) for $m = 7$ and $n = 5$. The patterns in this last figure can be used for determining such spanning subgraphs for these graphs for any odd numbers m and n. □

To conclude this section, we present the remaining open problem.

Problem 2.3.8.

(a) Is there a Hamilton cycle for the alphabet graph $X_{m,n}$ for $m = 5$ and any even n?

(b) Is there a spanning 2-connected subgraph with (at most) $|V| + 1$ edges for the alphabet graph $X_{m,n}$ for $m = 3$ and any even n?

Chapter 3

Ramsey Numbers for Paths Versus Wheels, Kipases or Fans

Abstract In this chapter we study the Ramsey numbers for paths versus wheels, kipases or fans. We determine the values of $R(P_n, W_m)$, $R(P_n, \hat{K}_m)$ and $R(P_n, F_m)$ for some values of n and m. We also give lower bounds and upper bounds for $R(P_n, W_m)$, $R(P_n, \hat{K}_m)$ and $R(P_n, F_m)$ for the other values of n and m.

3.1 Introduction

We recall that the *Ramsey number* $R(F, H)$ for two graphs F and H is defined as the smallest positive integer p such that every graph G on p vertices satisfies the following condition: G contains F as a subgraph or $\overline{G}$ contains H as a subgraph.

We study the Ramsey numbers for paths versus wheels, kipases or fans. The Ramsey numbers for paths versus wheels, for paths versus kipases and for paths versus fans are presented in Section 3.2, Section 3.3 and Section 3.4, respectively.

3.2 Path-wheel Ramsey numbers

In [44] we studied the Ramsey numbers for paths versus wheels. We determine the values of $R(P_n, W_m)$ for the following values of n and m: $n = 1, 2, 3$ or 5 and $m \geq 3$; $n = 4$ and $m = 3, 4, 5$ or 7; $n \geq 6$ and ($3 \leq$ odd $m \leq 2n-1$) or ($4 \leq$ even $m \leq n+1$); odd $n \geq 7$ and $m = 2n-2$ or $m = 2n$ or $m \geq (n-3)^2$; odd $n \geq 9$ and $q \cdot n - 2q + 1 \leq m \leq q \cdot n - q + 2$ with $3 \leq q \leq n-5$. Moreover, we give lower bounds and upper bounds for $R(P_n, W_m)$ for the other values of m and n. These results are presented in this section. The Ramsey numbers for 'small' paths versus wheels or the Ramsey numbers for paths versus 'small' wheels are given in Theorem 3.2.2. The Ramsey numbers for odd paths versus 'large' wheels are given in the corollary based on Lemma 3.2.3. In Corollary 3.2.5 and Theorem 3.2.6 we present lower bounds and upper bounds for $R(P_n, W_m)$ for (odd $n \geq 7$ and $q \cdot n - q + 3 \leq m \leq q \cdot n - 2q + n - 2$ with $2 \leq q \leq n-5$) or ($n \geq 6$ and $n+2 \leq$ even $m \leq 2n-4$) or (even $n \geq 4$ and $m = 2n-2$ or $m \geq 2n$).

Let us start with Lemma 3.2.1. This lemma plays a key role in the proofs for some lemmas or some theorems in this chapter.

Lemma 3.2.1. *Let $n \geq 3$ and G be a graph on at least n vertices containing no P_n. Let the paths $P^1, P^2, \ldots, P^k$ in G be chosen in the following way: $\bigcup_{j=1}^{k} V(P^j) = V(G)$, P^1 is a longest path in G, and, if $k > 1$, P^{i+1} is a longest path in $G - \bigcup_{j=1}^{i} V(P^j)$ for $1 \leq i \leq k-1$. Denote by ℓ_j the number of vertices on the path P^j. Let z be an end vertex of P^k. Then:*

- *(a) $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_k$;*
- *(b) If $\ell_k \geq \lfloor n/2 \rfloor$, then $N(z) \subset V(P^k)$;*
- *(c) If $\ell_k < \lfloor n/2 \rfloor$, then $|N(z)| \leq \lfloor n/2 \rfloor - 1$.*

Proof. (a) obviously follows from the choice of the paths. From this choice we can also deduce that for any integer x with $1 \leq x < k$, the number of neighbors of z in $V(P^x)$ is

$$\begin{cases} \leq \left\lfloor \frac{\ell_x + 1 - 2\ell_k}{2} \right\rfloor & \text{if } \ell_x \geq 2\ell_k + 1 \\ 0 & \text{if } \ell_x < 2\ell_k + 1. \end{cases} \tag{3.1}$$

This can be checked easily: First order the neighbors of z on P^x according to the order of their appearance on P^x in a fixed orientation. Then observe that between any two successive neighbors of z on P^x, there is at least one nonneighbor of z, while before the first and after the last neighbor of z on P^x, there are at least ℓ_k nonneighbors of z.

(b) Assume $\ell_k \geq \lfloor n/2 \rfloor$. Then $2\ell_k + 1 \geq n > \ell_1$. So by the above observation, we conclude that there is no neighbor of z in $V(G) \setminus V(P^k)$.

(c) Now assume $\ell_k < \lfloor n/2 \rfloor$. If z has no neighbors in $V(G) \setminus V(P^k)$, we are done. If z has some neighbors in $V(G) \setminus V(P^k)$, similar counting arguments as above yield the desired result: Denote by $h_1, \ldots, h_t$ the numbers of vertices on the paths $P^1, \ldots, P^k$ that contain a neighbor of z, chosen in such a way that $h_t \geq \ldots \geq h_1$, and denote by $d_1, \ldots, d_t$ the numbers of neighbors of z on the corresponding paths. Then, arguing as above, we obtain $h_1 = \ell_k \geq d_1 + 1$ and $h_2 \geq 2h_1 + 2d_2 - 1$. Similarly, observing that z connects any two of the considered paths, and using the same elementary counting techniques, we get, if $t \geq 3$, $h_j \geq 2(\frac{h_{j-1}-1}{2} + 2) + 2d_j - 1 = h_{j-1} + 2d_j + 2$ for $3 \leq j \leq t$. This implies, for $t \geq 2$, that $h_t \geq 2(d_1 + \ldots + d_t) + 2(t-2) + 1 \geq 2|N(z)| + 1$. Since $h_t \leq n - 1$ and $|N(z)|$ is an integer, this yields the desired result. □

Theorem 3.2.2.

$$R(P_n, W_m) = \begin{cases} 1 & \textit{for } n = 1 \textit{ and } m \geq 3 \\ m+1 & \textit{for either } n = 2 \textit{ and } m \geq 3 \\ & \textit{or } n = 3 \textit{ and even } m \geq 4 \\ m+2 & \textit{for } n = 3 \textit{ and odd } m \geq 5 \\ 3n-2 & \textit{for either } n = 3 \textit{ and } m = 3 \\ & \textit{or } n \geq 4 \textit{ and } 3 \leq \textit{odd } m \leq 2n-1 \\ 2n-1 & \textit{for } n \geq 4 \textit{ and } 4 \leq \textit{even } m \leq n+1. \end{cases}$$

Proof. The cases for which $n = 1$ or $n = 2$ are almost trivial and left to the reader. The rest of the proof we divide into three cases.

Case 1 $n = 3$ and $m \geq 4$.
The graph consisting of $\lfloor \frac{m+1}{2} \rfloor$ disjoint copies of K_2 shows that

$$R(P_3, W_m) > \begin{cases} m & \text{for even } m \\ m+1 & \text{for odd } m. \end{cases}$$

Now let G be a graph that contains no P_3 and has order

$$|V(G)| = \begin{cases} m+1 & \text{for even } m \\ m+2 & \text{for odd } m. \end{cases}$$

Since $|V(G)|$ is odd and G contains no P_3, there is a vertex $z \in V(G)$ with $|N(z)| = 0$. Since $G - z$ contains no P_3, the vertices of $V(G) \setminus \{z\}$ have degree at least $m-2$ in $\overline{G-z}$. This implies that there exists a cycle C_m in $\overline{G-z}$. Hence $\overline{G}$ contains a W_m.

Case 2 ($n = 3$ and $m = 3$) or ($n \geq 4$ and $3 \leq$ odd $m \leq 2n-1$).
The graph $3K_{n-1}$ shows that $R(P_n, W_m) > 3n-3$. Let G be a graph on $3n-2$ vertices and assume that G contains no P_n. We are going to show that $\overline{G}$ contains a W_m. Choose the paths $P^1, \ldots, P^k$ and the vertex z as in Lemma 3.2.1. Since $|V(G)| = 3n-2$, $\ell_k \leq n-2$. If $\ell_k < \lfloor n/2 \rfloor$ then, by Lemma 3.2.1(c), we obtain $|N(z)| \leq \lfloor n/2 \rfloor - 1 \leq n-3$. If $\lfloor n/2 \rfloor \leq \ell_k \leq n-2$ then, by Lemma 3.2.1(b), we obtain $|N(z)| \leq \ell_k - 1 \leq n-3$. Hence, $|N[z]| \leq n-2$. We are going to show that there is a W_m in $\overline{G}$ with z as a hub. We distinguish the following three subcases.

Subcase 2.1 $n \geq 3$ and $3 \leq$ odd $m < \lfloor (3n+1)/2 \rfloor$.
Then $|V(G) \setminus N[z]| \geq (3n-2) - (n-2) = 2n$. We can apply the result from [13] that $R(P_n, C_m) = 2n-1$ for $3 \leq$ odd $m \leq \lfloor (3n+1)/2 \rfloor$. This implies that $\overline{G - N[z]}$ contains a C_m. So, there is a W_m in $\overline{G}$ with z as a hub.

Subcase 2.2 $n \geq 4$ and $\lfloor (3n+1)/2 \rfloor \leq$ odd $m \leq 2n-1$ and $|N(z)| \leq \lfloor n/2 \rfloor - 1$.
Then $|V(G) \setminus N[z]| \geq (3n-2) - \lfloor n/2 \rfloor \geq 2n - 1 + \lfloor n/2 \rfloor - 1 \geq m + \lfloor n/2 \rfloor - 1$. We can apply the result from [13] that $R(P_n, C_m) = m + \lfloor n/2 \rfloor - 1$ for odd $m \geq \lfloor (3n+1)/2 \rfloor$. This implies that $\overline{G - N[z]}$ contains a C_m. So, there is a W_m in $\overline{G}$ with z as a hub.

Subcase 2.3 $n \geq 4$ and $\lfloor (3n+1)/2 \rfloor \leq$ odd $m \leq 2n-1$ and $|N(z)| \geq \lfloor n/2 \rfloor$.
By Lemma 3.2.1(b), we find $N(z) \subset V(P^k)$. Hence, $\ell_k \geq \lfloor n/2 \rfloor + 1$. Since $|V(G)| = 3n-2$ and $\ell_k \geq \lfloor n/2 \rfloor + 1$, $4 \leq k \leq 5$.

For $k = 5$ and $m = 3 \; mod \; 4$, take the first $\lceil m/4 \rceil$ vertices of P^1 (in some fixed orientation) and name them $u_1, \ldots, u_{\lceil m/4 \rceil}$, starting at an end vertex; take the first $\lceil m/4 \rceil$ vertices of P^2 (in some fixed orientation) and name them $v_1, \ldots, v_{\lceil m/4 \rceil}$, starting at an end vertex; take the first $\lceil m/4 \rceil$ vertices of P^3 (in some fixed orientation) and name them $w_1, \ldots, w_{\lceil m/4 \rceil}$, starting at an end vertex; take the first $\lfloor m/4 \rfloor$ vertices of P^4 (in some fixed orientation) and name them $x_1, \ldots, x_{\lfloor m/4 \rfloor}$, starting at an end vertex. Since P^1 is chosen as a longest path in G, it is obvious that $u_i v_i \notin E(G)$ for $i = 1, \ldots, \lceil m/4 \rceil$, $u_i x_{i+1} \notin E(G)$ for $i = 1, \ldots, \lfloor m/4 \rfloor - 1$, and $u_{\lfloor m/4 \rfloor} w_{\lceil m/4 \rceil} \notin E(G)$. Since P^2 is chosen as a longest path in $G - V(P^1)$, it is obvious that $v_i w_i \notin E(G)$ for $i = 1, \ldots, \lceil m/4 \rceil$. Since P^3 is chosen as a longest path in

$G - (V(P^1) \cup V(P^2))$, it is obvious that $w_i x_i \notin E(G)$ for $i = 1, \ldots, \lfloor m/4 \rfloor$. Since P^1 is chosen as a longest path in G, $\ell_4 \geq \lfloor n/2 \rfloor + 1$ and $m \leq 2n - 1$, it is obvious that $u_{\lceil m/4 \rceil} x_1 \notin E(G)$. So we can obtain a cycle C_m in $\overline{G}$, i.e., $x_1 w_1 v_1 u_1 x_2 w_2 v_2 u_2 \ldots x_{\lfloor m/4 \rfloor} w_{\lfloor m/4 \rfloor} v_{\lfloor m/4 \rfloor} u_{\lfloor m/4 \rfloor} w_{\lceil m/4 \rceil} v_{\lceil m/4 \rceil} u_{\lceil m/4 \rceil} x_1$. Hence, there is a W_m in $\overline{G}$ with z as a hub.

For $k = 5$ and $m = 1$ *mod* 4, take the first $\lfloor m/4 \rfloor$ vertices of P^1 (in some fixed orientation) and name them $u_1, \ldots, u_{\lfloor m/4 \rfloor}$, starting at an end vertex; take the other end vertex of P^1 and name it u_{ℓ_1}; take the first $\lfloor m/4 \rfloor$ vertices of P^2 (in some fixed orientation) and name them $v_1, \ldots, v_{\lfloor m/4 \rfloor}$, starting at an end vertex; take the first $\lfloor m/4 \rfloor$ vertices of P^3 (in some fixed orientation) and name them $w_1, \ldots, w_{\lfloor m/4 \rfloor}$, starting at an end vertex; take the first $\lfloor m/4 \rfloor$ vertices of P^4 (in some fixed orientation) and name them $x_1, \ldots, x_{\lfloor m/4 \rfloor}$, starting at an end vertex. Since P^1 is chosen as a longest path in G, it is obvious that $u_i v_i \notin E(G)$ for $i = 1, \ldots, \lfloor m/4 \rfloor$, $u_i x_{i+1} \notin E(G)$ for $i = 1, \ldots, \lfloor m/4 \rfloor - 1$, $u_{\lfloor m/4 \rfloor} x_{\lfloor m/4 \rfloor} \notin E(G)$, $u_{\ell_1} w_{\lfloor m/4 \rfloor} \notin E(G)$ and $u_{\ell_1} x_1 \notin E(G)$. Since P^2 is chosen as a longest path in $G - V(P^1)$, it is obvious that $v_i w_i \notin E(G)$ for $i = 1, \ldots, \lfloor m/4 \rfloor$. Since P^3 is chosen as a longest path in $G - (V(P^1) \cup V(P^2))$, it is obvious that $w_i x_i \notin E(G)$ for $i = 1, \ldots, \lfloor m/4 \rfloor - 1$. So we can obtain a cycle C_m in $\overline{G}$, i.e., $x_1 w_1 v_1 u_1 x_2 w_2 v_2 u_2 \ldots x_{\lfloor m/4 \rfloor - 1} w_{\lfloor m/4 \rfloor - 1} v_{\lfloor m/4 \rfloor - 1} u_{\lfloor m/4 \rfloor - 1} x_{\lfloor m/4 \rfloor} u_{\lfloor m/4 \rfloor} v_{\lfloor m/4 \rfloor} w_{\lfloor m/4 \rfloor} u_{\ell_1} x_1$. Hence, there is a W_m in $\overline{G}$ with z as a hub.

For $k = 4$, name the vertices of P^1 (in some fixed orientation, starting at an end vertex) $u_1, \ldots, u_{\ell_1}$; name the vertices of P^2 (in some fixed orientation, starting at an end vertex) $v_1, \ldots, v_{\ell_2}$; name the vertices of P^3 (in some fixed orientation, starting at an end vertex) $w_1, \ldots, w_{\ell_3}$. Since P^1 is chosen as a longest path in G, $\ell_1 \leq n-1$ and $\ell_3 \geq \lfloor n/2 \rfloor + 1$, it is obvious that $u_i v_i \notin E(G)$ for $i = 1, \ldots, \ell_2$, $u_i v_{i+1} \notin E(G)$ for $i = 1, \ldots, \ell_2 - 1$, $u_i w_{i+1} \notin E(G)$ for $i = 1, \ldots, \ell_3 - 1$, and $u_i w_1 \notin E(G)$ for $i = 1, \ldots, \ell_1$. Since P^2 is chosen as a longest path in $G - V(P^1)$ and $\ell_3 \geq \lfloor n/2 \rfloor + 1$, it is obvious that $v_i w_i \notin E(G)$ for $i = 1, \ldots, \ell_3$, and $v_i w_1 \notin E(G)$ for $i = 2, \ldots, \ell_2$. Since $\ell_1 + \ell_2 + \ell_3 + \ell_4 = 3n - 2$ and $\ell_1 + \ell_4 - \ell_2 \leq (n-1)$, $2\ell_2 + \ell_3 = (\ell_1 + \ell_2 + \ell_3 + \ell_4) - (\ell_1 + \ell_4 - \ell_2) \geq 2n - 1$. So we can obtain a cycle C_m for $m = 7, \ldots, 2\ell_2 + \ell_3$ in $\overline{G}$, i.e.,

- if $m = 3t - 2$ and $3 \leq t \leq \ell_3$, $C_m : w_1 v_1 u_1 w_2 v_2 u_2 \ldots w_{t-1} v_{t-1} u_{t-1} v_t w_1$;
- if $m = 3t - 1$ and $3 \leq t \leq \ell_3$, $C_m : w_1 v_1 u_1 w_2 v_2 u_2 \ldots w_{t-1} v_{t-1} u_{t-1} w_t v_t w_1$;
- if $m = 3t$ and $3 \leq t \leq \ell_3$, $C_m : w_1 v_1 u_1 w_2 v_2 u_2 \ldots w_{t-1} v_{t-1} u_{t-1} w_t v_t u_t w_1$;

- if $m = 3\ell_3 + 2t - 1$ and $1 \le t \le \ell_2 - \ell_3$, $C_m : w_1v_1u_1w_2v_2u_2 \ldots w_{\ell_3}v_{\ell_3}u_{\ell_3}v_{\ell_3+1}\ u_{\ell_3+1}v_{\ell_3+2}u_{\ell_3+2} \ldots v_{\ell_3+t-1}u_{\ell_3+t-1}v_{\ell_3+t}w_1$;
- if $m = 3\ell_3+2t$ and $1 \le t \le \ell_2-\ell_3$, $C_m : w_1v_1u_1w_2v_2u_2 \ldots w_{\ell_3}v_{\ell_3}u_{\ell_3}v_{\ell_3+1}\ u_{\ell_3+1}v_{\ell_3+2}u_{\ell_3+2} \ldots v_{\ell_3+t-1}u_{\ell_3+t-1}v_{\ell_3+t}u_{\ell_3+t}w_1$.

Hence, there is a W_m in $\overline{G}$ with z as a hub.

Case 3 $n \ge 4$ and $4 \le$ even $m \le n+1$.
The graph $2K_{n-1}$ shows that $R(P_n, W_m) > 2n-2$. Let G be a graph on $2n-1$ vertices and assume that G contains no P_n. We are going to show that $\overline{G}$ contains a W_m. Choose the paths $P^1, \ldots, P^k$ and the vertex z as in Lemma 3.2.1. Since $|V(G)| = 2n-1$ and G does not contain a P_n, $k \ge 3$ and $\ell_k \le \lfloor (2n-1)/3 \rfloor \le n-2$. By similar arguments as in the proof of Case 2, this implies that $|N(z)| \le n-3$. We are going to show that there is a W_m in $\overline{G}$ with z as a hub. We distinguish the following two subcases.

Subcase 3.1 $|N(z)| \le \lfloor n/2 \rfloor - 1$.
Then $|V(G) \setminus N[z]| \ge (2n-1) - \lfloor n/2 \rfloor \ge n + m/2 - 1$. We can apply the result from [13] that $R(P_n, C_m) = n + m/2 - 1$ for $4 \le$ even $m \le n+1$. This implies that $\overline{G - N[z]}$ contains a C_m. So, there is a W_m in $\overline{G}$ with z as a hub.

Subcase 3.2 $|N(z)| \ge \lfloor n/2 \rfloor$.
By Lemma 3.2.1(b), we find $N(z) \subset V(P^k)$. Hence, $\ell_k \ge \lfloor n/2 \rfloor + 1$. Since $|V(G)| = 2n-1$, $k = 3$. Take the first $m/2$ vertices of P^1 (in some fixed orientation) and name them $u_1, \ldots, u_{m/2}$, starting at an end vertex. Also take the first $m/2$ vertices of P^2 (in some fixed orientation) and name them $v_1, \ldots, v_{m/2}$, starting at an end vertex. Since P^1 is chosen as a longest path in G, it is obvious that $u_iv_i \notin E(G)$ for $i = 1, \ldots, m/2$, $u_iv_{i+1} \notin E(G)$ for $i = 1, \ldots, m/2-1$, and $u_{m/2}v_1 \notin E(G)$. So there is a W_m in $\overline{G}$ with z as a hub. □

The following lemma provides upper bounds that yield several path-wheel Ramsey numbers in the sequel.

Lemma 3.2.3. *If n is odd, $n \ge 5$ and $m \ge 2n-2$, then*

$$R(P_n, W_m) \le \begin{cases} m+n-1 & \text{for } m = 1 \ mod(n-1) \\ m+n-2 & \text{for other values of } m. \end{cases}$$

Proof. Let G be a graph that contains no P_n and has order

$$|V(G)| = \begin{cases} m+n-1 & \text{for } m = 1 \ mod(n-1) \\ m+n-2 & \text{for other values of } m. \end{cases} \quad (3.2)$$

Choose the paths $P^1, \dots, P^k$ and the vertex z in G as in Lemma 3.2.1. Because of (3.2), not all P^i can have $n-1$ vertices, so $\ell_k \le n-2$. By similar arguments as in the proof of Case 2 of Theorem 3.2.2, this implies that $|N(z)| \le n-3$. Hence, z is not a neighbor of (at least) $(m+n-2)-1-(n-3) = m$ vertices. We will use the following result that has been proved in [13]: $R(P_n, C_m) = m + \lfloor n/2 \rfloor - 1$ for $m \ge \lfloor (3n+1)/2 \rfloor$. We distinguish the following cases.

Case 1 $|N(z)| \le \lfloor n/2 \rfloor - 1$
Since $|V(G) \setminus N[z]| \ge m + \lfloor n/2 \rfloor - 1$, we find that $\overline{G - N[z]}$ contains a C_m. So, there is a W_m in $\overline{G}$ with z as a hub.

Case 2 Suppose that there is no choice for P^k and z such that Case 1 applies. Then $|N(w)| \ge \lfloor n/2 \rfloor$ for any end vertex w of a path on ℓ_k vertices in $G - \bigcup_{j=1}^{k-1} V(P^j)$. This implies that all neighbors of such w are in $V(P^k)$ and $\ell_k \ge \lfloor n/2 \rfloor + 1$. So for the two end vertices z_1 and z_2 of P^k we have that $|N(z_i) \cap V(P^k)| \ge \lfloor n/2 \rfloor \ge \ell_k/2$. Let $P^k : z_1 = v_1 v_2 \dots v_{\ell_k} = z_2$. Then, by standard arguments in hamiltonian graph theory, we can find an index $i \in \{2, \dots, \ell_k - 1\}$ such that $z_1 v_{i+1}$ and $z_2 v_i$ are edges of G. It is clear that we can find a cycle on ℓ_k vertices in G. This implies that any vertex of $V(P^k)$ could serve as w. By the assumption of this last case, we conclude that there are no edges in G between $V(P^k)$ and the other vertices. This also implies that all vertices of P^k have degree in $\overline{G}$ at least

$$\begin{cases} m+1 & \text{if } |V(G)| = m+n-1 \\ m & \text{if } |V(G)| = m+n-2. \end{cases}$$

We now turn to P^{k-1} and consider one of its end vertices w. Since $\ell_{k-1} \ge \ell_k \ge \lfloor n/2 \rfloor + 1$, similar arguments as in the proof of Lemma 3.2.1 show that all neighbors of w are on P^{k-1}. If $|N(w)| < \lfloor n/2 \rfloor$, we get a W_m in $\overline{G}$ as in Case 1. So we may assume $|N(w_i) \cap V(P^{k-1})| \ge \lfloor n/2 \rfloor \ge \ell_{k-1}/2$ for both end vertices w_1 and w_2 of P^{k-1}. By similar arguments as before we obtain a cycle on ℓ_{k-1} vertices in G. This implies that any vertex of $V(P^{k-1})$ could serve as w. By the assumption of this last case, we conclude that there are no edges in G between $V(P^{k-1})$ and the other vertices. This also implies that all vertices of P^{k-1} have degree in $\overline{G}$ at least

$$\begin{cases} m & \text{if } |V(G)| = m+n-1 \\ m-1 & \text{if } |V(G)| = m+n-2. \end{cases} \tag{3.3}$$

(Note that P^{k-1} can have $n-1$ vertices, whereas $\ell_k \le n-2$.)

Repeating the above arguments for $P^{k-2}, \dots, P^1$ we eventually conclude that all vertices of G have degree in $\overline{G}$ at least as in (3.3).

Now let $H = \overline{G} - V(P^k)$. If $|V(G)| = m+n-1$, then all vertices in $V(H)$ have degree at least $m - \ell_k \geq m/2 + (n-1) - \ell_k \geq \frac{1}{2}(m + 2n - 2 - \ell_k - (n-2)) = \frac{1}{2}(m + n - \ell_k) = \frac{1}{2}(|V(H)| + 1)$. By a standard result in hamiltonian graph theory this implies that H is *pancyclic*, i.e., it contains cycles of every length from 3 up to $|V(H)|$ (see e.g. [11] Corollary 4.31). In particular, H contains a C_m, hence $\overline{G}$ contains a W_m with z as a hub. If $V(G) = m + n - 2$, then all vertices in $V(H)$ have degree at least $m - 1 - \ell_k \geq m/2 + (n-1) - 1 - \ell_k \geq \frac{1}{2}(m + 2n - 4 - \ell_k - (n-2)) = \frac{1}{2}(m + n - 2 - \ell_k) = \frac{1}{2}|V(H)|$. This implies that H is pancyclic unless H is a complete bipartite graph $K_{p,p}$ with $p = \frac{1}{2}|V(H)|$ (see e.g. [11] Corollary 4.31). In the first case we get a W_m in $\overline{G}$ as before. In the latter case, if $|V(H)| = m$ we also obtain a W_m in $\overline{G}$; if $|V(H)| \geq m + 1$, then note that $G \supset \overline{H} \supset K_p \supset P_p$. By our assumptions this implies that $p \leq n - 1$, while on the other hand $p \geq \frac{1}{2}(m + 1)$, so $\frac{1}{2}(m + 1) \leq n - 1$ or $m \leq 2n - 3$, contradicting that $m \geq 2n - 2$. This completes the proof of Lemma 3.2.3. □

Corollary 3.2.4. *If ($n = 5$ and $m = 8$ or $m \geq 10$) or (n is odd, $n \geq 7$ and $m = 2n - 2$ or $m = 2n$ or $m \geq (n-3)^2$) or (n is odd, $n \geq 9$ and $q \cdot n - 2q + 1 \leq m \leq q \cdot n - q + 2$ with $3 \leq q \leq n - 5$), then*

$$R(P_n, W_m) = \begin{cases} m + n - 1 & \text{for } m = 1 \ mod(n-1) \\ m + n - 2 & \text{for other values of } m. \end{cases}$$

Proof. Let r denote the remainder of m divided by $n - 1$, so $m = p(n-1) + r$ for some $0 \leq r \leq n - 2$. Then for ($n = 5$ and $m = 8$ or $m \geq 10$) or (odd $n \geq 7$ and $m = 2n - 2$ or $m = 2n$ or $m \geq (n-3)^2$) or ($n \geq 9$ and $q \cdot n - 2q + 1 \leq m \leq q \cdot n - q + 2$ with $3 \leq q \leq n - 5$) the graphs

$$\begin{cases} (p-1)K_{n-1} \cup 2K_{n-2} & \text{for } r = 0 \\ (p+1)K_{n-1} & \text{for } r = 1 \text{ or } 2 \\ (p + r + 1 - n)K_{n-1} \cup (n + 1 - r)K_{n-2} & \text{for other values of } r \end{cases}$$

show that

$$R(P_n, W_m) > \begin{cases} m + n - 2 & \text{for } m = 1 \ mod(n-1) \\ m + n - 3 & \text{for other values of } m. \end{cases}$$

Lemma 3.2.3 completes the proof. □

Corollary 3.2.5. *If n is odd, $n \geq 7$ and $q \cdot n - q + 3 \leq m \leq q \cdot n - 2q + n - 2$ with $2 \leq q \leq n-5$, then*

$$m+n-2 \geq R(P_n, W_m) \geq max\left\{\left\lfloor\frac{m}{n-1}\right\rfloor(n-1)+n,\ m+\left\lfloor\frac{m-1}{\lceil m/(n-1)\rceil}\right\rfloor\right\}.$$

Proof. Let $t = \left\lceil\frac{m}{n-1}\right\rceil$ and let s denote the remainder of $m-1$ divided by t. Then for m and n satisfying $\left\lfloor\frac{m}{n-1}\right\rfloor(n-1)+n \geq m+\left\lfloor\frac{m-1}{t}\right\rfloor$, the graph tK_{n-1} shows that $R(P_n, W_m) > \left\lfloor\frac{m}{n-1}\right\rfloor(n-1)+n-1$.

For other values of m and n, the graph $sK_{\lceil(m-1)/t\rceil} \cup (t-s+1)K_{\lfloor(m-1)/t\rfloor}$ shows that $R(P_n, W_m) > m-1+\left\lfloor\frac{m-1}{\lceil m/(n-1)\rceil}\right\rfloor$.

The upper bound comes from Lemma 3.2.3. □

Theorem 3.2.6. *If ($n \geq 6$ and m is even, $n+2 \leq m \leq 2n-4$) or (n is even, $n \geq 4$ and $m = 2n-2$ or $m \geq 2n$), then*

$$m+\lfloor 3n/2\rfloor - 2 \geq R(P_n, W_m) \geq$$

$$max\left\{\left\lfloor\frac{m-1}{n-1}\right\rfloor(n-1)+n,\ m+\left\lfloor\frac{m-1}{\lceil(m-1)/(n-1)\rceil}\right\rfloor\right\}.$$

Proof. Let $t = \left\lceil\frac{m-1}{n-1}\right\rceil$ and let s denote the remainder of $m-1$ divided by t. Then for m and n satisfying $\left\lfloor\frac{m-1}{n-1}\right\rfloor(n-1)+n \geq m+\left\lfloor\frac{m-1}{t}\right\rfloor$, the graph tK_{n-1} shows that $R(P_n, W_m) > \left\lfloor\frac{m-1}{n-1}\right\rfloor(n-1)+n-1$.

For other values of m and n, the graph $sK_{\lceil(m-1)/t\rceil} \cup (t-s+1)K_{\lfloor(m-1)/t\rfloor}$ shows that $R(P_n, W_m) > m-1+\left\lfloor\frac{m-1}{\lceil(m-1)/(n-1)\rceil}\right\rfloor$.

Let G be a graph on $m+\lfloor 3n/2\rfloor-2$ vertices, and assume that G contains no P_n. Choose the paths $P^1, \ldots, P^k$ and the vertex z in G as in Lemma 3.2.1. Since $\ell_k \leq n-1$ and by similar arguments as in the proof of Case 2 of Theorem 3.2.2, $|N(z)| \leq n-2$. Hence, $|V(G) \setminus N[z]| \geq m+\lfloor n/2\rfloor-1$. We can apply the result from [13] that $R(P_n, C_m) = m+\lfloor n/2\rfloor-1$ for (even $m \geq n \geq 2$) or ($n \geq 4$ and $m \geq 3n/2$). This implies that $\overline{G-N[z]}$ contains a C_m. So, there is a W_m in $\overline{G}$ with z as a hub. □

3.3 Path-kipas Ramsey numbers

We studied in [45] the Ramsey numbers for paths versus kipases. We determine the Ramsey numbers $R(P_n, \hat{K}_m)$ for the following values of n and m: $1 \leq n \leq 5$ and $m \geq 3$; $n \geq 6$ and $3 \leq$ odd $m \leq 2n-1$ or $4 \leq$ even $m \leq n+1$; $6 \leq n \leq 7$ and $m = 2n-2$ or $m \geq 2n$; $n \geq 8$ and $m = 2n-2$ or $m = 2n$ or $(q \cdot n - 2q + 1 \leq m \leq q \cdot n - q + 2$ with $3 \leq q \leq n-5)$ or $m \geq (n-3)^2$; odd $n \geq 9$ and $(q \cdot n - 3q + 1 \leq m \leq q \cdot n - 2q$ with $3 \leq q \leq (n-3)/2)$ or $(q \cdot n - q - n + 4 \leq m \leq q \cdot n - 2q$ with $(n-1)/2 \leq q \leq n-4)$. These results are presented in this section. The Ramsey numbers for 'small' paths versus kipases or paths versus 'small' kipases are given in Corollary 3.3.1. The Ramsey numbers for paths versus 'large' kipases are given in Corollary 3.3.3 and Corollary 3.3.5. Moreover, we also give lower bounds and upper bounds for $R(P_n, \hat{K}_m)$ for (odd $n \geq 11$ and $q \cdot n - q + 3 \leq m \leq q \cdot n - 3q + n - 3$ with $2 \leq q \leq (n-7)/2)$ or (even $n \geq 8$ and $q \cdot n - q + 3 \leq m \leq q \cdot n - 2q + n - 2$ with $2 \leq q \leq n-5)$ or $(n \geq 6$ and $n+2 \leq$ even $m \leq 2n-4)$ in Corollary 3.3.6, Corollary 3.3.7 and Theorem 3.3.8.

Corollary 3.3.1.

$$R(P_n, \hat{K}_m) = \begin{cases} 1 & \textit{for } n = 1 \textit{ and } m \geq 3 \\ m+1 & \textit{for either } n = 2 \textit{ and } m \geq 3 \\ & \textit{or } n = 3 \textit{ and even } m \geq 4 \\ m+2 & \textit{for } n = 3 \textit{ and odd } m \geq 5 \\ 3n-2 & \textit{for either } n = 3 \textit{ and } m = 3 \\ & \textit{or } n \geq 4 \textit{ and } 3 \leq \textit{odd } m \leq 2n-1 \\ 2n-1 & \textit{for } n \geq 4 \textit{ and } 4 \leq \textit{even } m \leq n+1. \end{cases}$$

Proof. The graphs

$$\begin{cases} P_1 & \text{for } n = 1 \text{ and } m \geq 3 \\ mP_1 & \text{for } n = 2 \text{ and } m \geq 3 \\ \left\lfloor \frac{m+1}{2} \right\rfloor K_2 & \text{for } n = 3 \text{ and } m \geq 4 \\ 3K_{n-1} & \text{for either } n = 3 \text{ and } m = 3 \\ & \text{or } n \geq 4 \text{ and } 3 \leq \text{odd } m \leq 2n-1 \\ 2K_{n-1} & \text{for } n \geq 4 \text{ and } 4 \leq \text{even } m \leq n+1 \end{cases}$$

give the best lower bounds for $R(P_n, \hat{K}_m)$ for the values of m and n in Corollary 3.3.1. Since $\hat{K}_m$ is a subgraph of W_m, Theorem 3.2.2 completes the proof. $\square$

Lemma 3.3.2 and Lemma 3.3.4 provide upper bounds that yield several Ramsey numbers in the sequel.

Lemma 3.3.2. *If $n \geq 4$ and $m \geq 2n-2$, then*

$$R(P_n, \hat{K}_m) \leq \begin{cases} m+n-1 & \textit{for } m = 1 \ mod(n-1) \\ m+n-2 & \textit{for other values of } m. \end{cases}$$

Proof. Let G be a graph that contains no P_n and has order

$$|V(G)| = \begin{cases} m+n-1 & \text{for } m = 1 \ mod(n-1) \\ m+n-2 & \text{for other values of } m. \end{cases} \tag{3.4}$$

Choose the paths $P^1, \ldots, P^k$ and the vertex z in G as in Lemma 3.2.1. Because of (3.4), not all P^i can have $n-1$ vertices, so $\ell_k \leq n-2$. If $\ell_k < \lfloor n/2 \rfloor$ then, by Lemma 3.2.1(c), we obtain $|N(z)| \leq \lfloor n/2 \rfloor - 1 \leq n-3$. If $\lfloor n/2 \rfloor \leq \ell_k \leq n-2$ then, by Lemma 3.2.1(b), we obtain $|N(z)| \leq \ell_k - 1 \leq n-3$. Hence, $|N[z]| \leq n-2$. We will use the following result that has been proved in [13]: $R(P_t, C_s) = s + \lfloor t/2 \rfloor - 1$ for $s \geq \lfloor (3t+1)/2 \rfloor$. We distinguish the following cases.

Case 1 $|N(z)| \leq \lfloor n/2 \rfloor - 2$ or n is odd and $|N(z)| = \lfloor n/2 \rfloor - 1$.
Since $|V(G) \setminus N[z]| \geq m + \lfloor n/2 \rfloor - 1$, we find that $\overline{G - N[z]}$ contains a C_m. So, there is a $\hat{K}_m$ in $\overline{G}$ with z as a hub.

Case 2 n is even and $|N(z)| = n/2 - 1$.
Since $|V(G) \setminus N[z]| \geq (m+n-2) - n/2 = m + n/2 - 2$, we find that $\overline{G - N[z]}$ contains a C_{m-1}; denote its vertices by $v_1, v_2, v_3, \ldots, v_{m-1}$ in the order of appearance on the cycle with a fixed orientation. There are $n/2-1$ vertices in $U = V(G) \setminus (V(C_{m-1}) \cup N[z])$, say $u_1, u_2, \ldots, u_{n/2-1}$. If some vertex v_i $(i = 1, \ldots, m-1)$ is no neighbor of some vertex u_j $(j = 1, \ldots, n/2-1)$, w.l.o.g. assume $v_{m-1}u_1 \notin E(G)$. Then $\overline{G}$ contains a $\hat{K}_m$ with z as a hub and its other vertices $v_1, v_2, v_3, \ldots, v_{m-2}, v_{m-1}, u_1$. Now let us assume that each of the v_i is adjacent to all u_j in G. For every choice of a subset of $n/2$ vertices from $V(C_{m-1})$, there is a path on $n-1$ vertices in G alternating between the vertices of this subset and the vertices of U, starting and terminating in two arbitrary vertices from the subset. Since G contains no P_n, there are no edges $v_iv_j \in E(G)$ for $i, j \in \{1, \ldots, m-1\}$. This implies that $V(C_{m-1}) \cup \{z\}$ induces a K_m in $\overline{G}$. Since G contains no P_n, no v_i is adjacent to a vertex of $N(z)$. This implies that $\overline{G}$ contains a $K_{m+1} - zw$ for any vertex $w \in N(z)$, and hence $\overline{G}$ contains a $\hat{K}_m$ with one of the v_i as a hub.

Case 3 Suppose that there is no choice for P^k and z such that one of the former cases applies. Then $|N(w)| \geq \lfloor n/2 \rfloor$ for any end vertex w of a path on ℓ_k vertices in $G - \bigcup_{j=1}^{k-1} V(P^j)$. This implies that all neighbors of such w are in $V(P^k)$ and $\ell_k \geq \lfloor n/2 \rfloor + 1$. So for the two end vertices z_1 and z_2 of P^k we have that $|N(z_i) \cap V(P^k)| \geq \lfloor n/2 \rfloor \geq \ell_k/2$. By standard arguments in hamiltonian graph theory, we can find an index $i \in \{2, \ldots, \ell_k - 1\}$ such that $z_1 v_{i+1}$ and $z_2 v_i$ are edges of G. It is clear that we can find a cycle on ℓ_k vertices in G. This implies that any vertex of $V(P^k)$ could serve as w. By the assumption of this last case, we conclude that there are no edges in G between $V(P^k)$ and the other vertices. This also implies that all vertices of P^k have degree at least m in $\overline{G}$.

We now turn to P^{k-1} and consider one of its end vertices w. Since $\ell_{k-1} \geq \ell_k \geq \lfloor n/2 \rfloor + 1$, similar arguments as in the proof of Lemma 3.2.1 show that all neighbors of w are on P^{k-1}. If $|N(w)| < \lfloor n/2 \rfloor$, we get a $\hat{K}_m$ in $\overline{G}$ as in Case 1 or Case 2. So we may assume $|N(w_i) \cap V(P^{k-1})| \geq \lfloor n/2 \rfloor \geq \ell_{k-1}/2$ for both end vertices w_1 and w_2 of P^{k-1}. By similar arguments as before we obtain a cycle on ℓ_{k-1} vertices in G. This implies that any vertex of $V(P^{k-1})$ could serve as w. By the assumption of this last case, we conclude that there are no edges in G between $V(P^{k-1})$ and the other vertices. This also implies that all vertices of P^{k-1} have degree at least $m-1$ in $\overline{G}$. (Note that P^{k-1} can have $n-1$ vertices, whereas $\ell_k \leq n-2$.)

Repeating the above arguments for $P^{k-2}, \ldots, P^1$ we eventually conclude that all vertices of G have degree at least $m-1$ in $\overline{G}$. Now let $H = \overline{G} - V(P^k)$. Then all vertices in $V(H)$ have degree at least $m - 1 - \ell_k \geq m/2 + (n-1) - 1 - \ell_k \geq \frac{1}{2}(m + 2n - 4 - \ell_k - (n-2)) = \frac{1}{2}(m + n - 2 - \ell_k) = \frac{1}{2}(|V(H)| - 1)$. Hence, there exists a Hamilton path in H. Since $|V(H)| \geq m$ and z is a neighbor of all vertices in H (in $\overline{G}$), it is clear that $\overline{G}$ contains a $\hat{K}_m$ with z as a hub. This completes the proof of Lemma 3.3.2. □

Corollary 3.3.3. *If* $(4 \leq n \leq 6$ *and* $m = 2n-2$ *or* $m \geq 2n)$ *or* $(n \geq 7$ *and* $m = 2n-2$ *or* $m = 2n$ *or* $m \geq (n-3)^2)$ *or* $(n \geq 8$ *and* $q \cdot n - 2q + 1 \leq m \leq q \cdot n - q + 2$ *with* $3 \leq q \leq n-5)$, *then*

$$R(P_n, \hat{K}_m) = \begin{cases} m + n - 1 & \text{for } m = 1 \bmod (n-1) \\ m + n - 2 & \text{for other values of } m. \end{cases}$$

Proof. Let r denote the remainder of m divided by $n-1$, so $m = p(n-1) + r$ for some $0 \leq r \leq n-2$. Then for $(4 \leq n \leq 6$ and $m = 2n-2$ or $m \geq 2n)$

or ($n \geq 7$ and $m = 2n-2$ or $m = 2n$ or $m \geq (n-3)^2$) or ($n \geq 8$ and $q \cdot n - 2q + 1 \leq m \leq q \cdot n - q + 2$ for $3 \leq q \leq n-5$), the graphs

$$\begin{cases} (p-1)K_{n-1} \cup 2K_{n-2} & \text{for } r = 0 \\ (p+1)K_{n-1} & \text{for } r = 1 \text{ or } 2 \\ (p+r+1-n)K_{n-1} \cup (n+1-r)K_{n-2} & \text{for other values of } r \end{cases}$$

show that

$$R(P_n, \hat{K}_m) > \begin{cases} m+n-2 & \text{for } m = 1 \; mod(n-1) \\ m+n-3 & \text{for other values of } m. \end{cases}$$

Lemma 3.3.2 completes the proof. □

Lemma 3.3.4. *If n is odd, $n \geq 7$ and $q \cdot n - q + 3 \leq m \leq q \cdot n - 2q + n - 2$ with $2 \leq q \leq n-5$, then $R(P_n, \hat{K}_m) \leq m + n - 3$.*

Proof. The proof is modeled along the lines of the proof of Lemma 3.3.2. Let G be a graph on $m+n-3$ vertices, and assume that G contains no P_n. We will show that $\overline{G}$ contains a $\hat{K}_m$. Choose the paths $P^1, \ldots, P^k$ and the vertex z in G as in Lemma 3.2.1. Since $|V(G)| = m+n-3$ with $n \geq 7$ and $q \cdot n - q + 3 \leq m \leq q \cdot n - 2q + n - 2$ with $2 \leq q \leq n-5$, $k \geq q+2$, and therefore not all P^i can have more than $n-3$ vertices. So $\ell_k \leq n-3$. By similar arguments as in the proof of Lemma 3.3.2, this implies that $|N(z)| \leq n-4$. We will use the following result that has been proved in [13]: $R(P_t, C_s) = s + \lfloor t/2 \rfloor - 1$ for $s \geq \lfloor (3t+1)/2 \rfloor$. We distinguish the following cases.

Case 1 $|N(z)| \leq \lfloor n/2 \rfloor - 2$.
Since $|V(G) \setminus N[z]| \geq m + \lfloor n/2 \rfloor - 1$, we find that $\overline{G - N[z]}$ contains a C_m. So, there is a $\hat{K}_m$ in $\overline{G}$ with z as a hub.

Case 2 $|N(z)| = \lfloor n/2 \rfloor - 1$.
Since $|V(G) \setminus N[z]| = (m+n-3) - \lfloor n/2 \rfloor = m + \lfloor n/2 \rfloor - 2$, we find that $\overline{G - N[z]}$ contains a C_{m-1}; denote its vertices by $v_1, v_2, v_3, \ldots, v_{m-1}$ in the order of appearance on the cycle with a fixed orientation. There are $\lfloor n/2 \rfloor - 1$ vertices in $U = V(G) \setminus (V(C_{m-1}) \cup N[z])$, say $u_1, u_2, \ldots, u_{\lfloor n/2 \rfloor - 1}$. If some vertex v_i ($i = 1, \ldots, m-1$) is no neighbor of some vertex u_j ($j = 1, \ldots, \lfloor n/2 \rfloor - 1$), w.l.o.g. assume $v_{m-1}u_1 \notin E(G)$. Then $\overline{G}$ contains a $\hat{K}_m$ with z as a hub and its other vertices $v_1, v_2, v_3, \ldots, v_{m-2}, v_{m-1}, u_1$. Now let us assume that each of the v_i is adjacent to all u_j in G. For every choice of a subset of $\lfloor n/2 \rfloor$ vertices

from $V(C_{m-1})$, there is a path on $n-2$ vertices in G alternating between the vertices of this subset and the vertices of U, starting and terminating in two arbitrary vertices from the subset. Let $z_1 \in N(z)$. Since G contains no P_n, there are no edges $v_i z \in E(G)$ and $v_i z_1 \in E(G)$ ($i \in \{1, \ldots, m-1\}$) and there is at most one edge $v_i v_j \in E(G)$ (for some $i, j \in \{1, \ldots, m-1\}$). Assume (at most) $v_1 v_2 \in E(G)$. This implies that $\overline{G}$ contains a $\hat{K}_m$ with hub v_{m-1} and its other vertices $v_1, z, v_2, z_1, v_3, \ldots, v_{m-4}, v_{m-3}, v_{m-2}$.

Case 3 Suppose that there is no choice for P^k and z such that one of the former cases applies. Then $|N(w)| \geq \lfloor n/2 \rfloor$ for any end vertex w of a path on ℓ_k vertices in $G - \bigcup_{j=1}^{k-1} V(P^j)$. This implies that all neighbors of such w are in $V(P^k)$ and $\ell_k \geq \lfloor n/2 \rfloor + 1$. So for the two end vertices z_1 and z_2 of P^k we have that $|N(z_i) \cap V(P^k)| \geq \lfloor n/2 \rfloor \geq \ell_k/2$. By similar arguments as in the proof of Lemma 3.3.2 we obtain a cycle on ℓ_k vertices in G. This implies that any vertex of $V(P^k)$ could serve as w. By the assumption of this last case, we conclude that there are no edges in G between $V(P^k)$ and the other vertices. This also implies that all vertices of P^k have degree at least m in $\overline{G}$.

We now turn to P^{k-1} and consider one of its end vertices w. Since $\ell_{k-1} \geq \ell_k \geq \lfloor n/2 \rfloor + 1$, similar arguments as in the proof of Lemma 3.2.1 show that all neighbors of w are on P^{k-1}. If $|N(w)| < \lfloor n/2 \rfloor$, we get a $\hat{K}_m$ in $\overline{G}$ as in Case 1 or Case 2. So we may assume $|N(w_i) \cap V(P^{k-1})| \geq \lfloor n/2 \rfloor \geq \ell_{k-1}/2$ for both end vertices w_1 and w_2 of P^{k-1}. By similar arguments as before we obtain a cycle on ℓ_{k-1} vertices in G. This implies that any vertex of $V(P^{k-1})$ could serve as w. By the assumption of this last case, we conclude that there are no edges in G between $V(P^{k-1})$ and the other vertices. This also implies that all vertices of P^{k-1} have degree at least $m-2$ in $\overline{G}$. (Note that P^{k-1} can have $n-1$ vertices, whereas $\ell_k \leq n-3$.)

Repeating the above arguments for $P^{k-2}, \ldots, P^1$ we eventually conclude that all vertices of G have degree at least $m-2$ in $\overline{G}$. Now let $H = \overline{G} - V(P^k)$. Then all vertices in $V(H)$ have degree at least $m - 2 - \ell_k \geq m/2 + n - 2 - \ell_k \geq \frac{1}{2}(m + 2n - 4 - \ell_k - (n-3)) = \frac{1}{2}(m + n - 1 - \ell_k) = \frac{1}{2}(|V(H)| + 2)$. This implies that there exists a Hamilton cycle in H. Since $|V(H)| \geq m$ and z is a neighbor of all vertices in H (in $\overline{G}$), it is clear that $\overline{G}$ contains a $\hat{K}_m$ with z as a hub. This completes the proof of Lemma 3.3.4. □

Corollary 3.3.5. *If* ($n = 7$ *and* $m = 15$) *or* (n *is odd,* $n \geq 9$ *and* ($q \cdot n - 3q + 1 \leq m \leq q \cdot n - 2q$ *with* $3 \leq q \leq (n-3)/2$) *or* ($q \cdot n - q - n + 4 \leq m \leq q \cdot n - 2q$ *with* $(n-1)/2 \leq q \leq n-4$)), *then* $R(P_n, \hat{K}_m) = m + n - 3$.

Proof. For $n = 7$ and $m = 15$, the graph $3K_6$ and for odd $n \geq 9$ and $m = q \cdot n - 2q - j$ with either ($3 \leq q \leq (n-3)/2$ and $0 \leq j \leq q-1$) or ($(n-1)/2 \leq q \leq n-5$ and $0 \leq j \leq n-q-4$), the graph $(q-j-1)K_{n-2} \cup (j+2)K_{n-3}$ shows that $R(P_n, \hat{K}_m) > m + n - 4$. Lemma 3.3.4 completes the proof. □

Corollary 3.3.6. *If n is odd, $n \geq 11$ and $q \cdot n - q + 3 \leq m \leq q \cdot n - 3q + n - 3$ with $2 \leq q \leq (n-7)/2$, then*

$$m+n-3 \geq R(P_n, \hat{K}_m) \geq max\left\{\left\lfloor \frac{m}{n-1} \right\rfloor (n-1) + n,\ m + \left\lfloor \frac{m-1}{\lceil m/(n-1) \rceil} \right\rfloor \right\}.$$

Proof. Let $t = \left\lceil \frac{m}{n-1} \right\rceil$ and let s denote the remainder of $m-1$ divided by t. Then for m and n satisfying $\left\lfloor \frac{m}{n-1} \right\rfloor (n-1) + n \geq m + \left\lfloor \frac{m-1}{t} \right\rfloor$, the graph tK_{n-1} shows that $R(P_n, F_m) > \left\lfloor \frac{m}{n-1} \right\rfloor (n-1) + n - 1$.

For other values of m and n, the graph $sK_{\lceil (m-1)/t \rceil} \cup (t-s+1)K_{\lfloor (m-1)/t \rfloor}$ shows that $R(P_n, F_m) > m - 1 + \left\lfloor \frac{m-1}{\lceil m/(n-1) \rceil} \right\rfloor$.

The upper bound comes from Lemma 3.3.4. □

Corollary 3.3.7. *If n is even, $n \geq 8$ and $q \cdot n - q + 3 \leq m \leq q \cdot n - 2q + n - 2$ with $2 \leq q \leq n-5$, then*

$$m+n-2 \geq R(P_n, \hat{K}_m) \geq max\left\{\left\lfloor \frac{m}{n-1} \right\rfloor (n-1) + n,\ m + \left\lfloor \frac{m-1}{\lceil m/(n-1) \rceil} \right\rfloor \right\}.$$

Proof. Let $t = \left\lceil \frac{m}{n-1} \right\rceil$ and let s denote the remainder of $m-1$ divided by t. Then for m and n satisfying $\left\lfloor \frac{m}{n-1} \right\rfloor (n-1) + n \geq m + \left\lfloor \frac{m-1}{t} \right\rfloor$, the graph tK_{n-1} shows that $R(P_n, \hat{K}_m) > \left\lfloor \frac{m}{n-1} \right\rfloor (n-1) + n - 1$.

For other values of m and n, the graph $sK_{\lceil (m-1)/t \rceil} \cup (t-s+1)K_{\lfloor (m-1)/t \rfloor}$ shows that $R(P_n, \hat{K}_m) > m - 1 + \left\lfloor \frac{m-1}{\lceil m/(n-1) \rceil} \right\rfloor$.

The upper bound comes from Lemma 3.3.2. □

Theorem 3.3.8. *If $n \geq 6$ and m is even with $n+2 \leq m \leq 2n-4$, then*

$$m + \left\lfloor \frac{3n}{2} \right\rfloor - 2 \geq R(P_n, \hat{K}_m) \geq \begin{cases} 2n-1 & \text{for } n+2 \leq m \leq n + \lfloor n/3 \rfloor \\ \frac{3m}{2} - 1 & \text{for } n + \lfloor n/3 \rfloor < m \leq 2n-4. \end{cases}$$

Proof. For $n \geq 6$ and $n+2 \leq$ even $m \leq n + \lfloor n/3 \rfloor$, the graph $2K_{n-1}$ shows that $R(P_n, \hat{K}_m) > 2n-2$. For $n \geq 6$ and $n + \lfloor n/3 \rfloor <$ even $m \leq 2n-4$, the graph $K_{m/2} \cup 2K_{m/2-1}$ shows that $R(P_n, \hat{K}_m) > \frac{3m}{2} - 2$.

Let G be a graph on $m + \lfloor 3n/2 \rfloor - 2$ vertices, and assume that G contains no P_n. Choose the paths $P^1, \ldots, P^k$ and the vertex z in G as in Lemma 3.2.1. By Lemma 3.2.1, $|N(z)| \leq n-2$. Hence, $|V(G) \setminus N[z]| \geq m + \lfloor n/2 \rfloor - 1$. We can apply the result from [13] that $R(P_n, C_m) = m + \lfloor n/2 \rfloor - 1$ for $2 \leq n \leq$ even m. This implies that $\overline{G - N[z]}$ contains a C_m. So, there is a $\hat{K}_m$ in $\overline{G}$ with z as a hub (there is even a wheel on $m+1$ vertices). □

3.4 Path-fan Ramsey numbers

We studied in [43] the Ramsey numbers for paths versus fans. We determine the Ramsey numbers $R(P_n, F_m)$ for the following values of n and m: $1 \leq n \leq 5$ and $m \geq 2$; $n \geq 6$ and $2 \leq m \leq (n+1)/2$; $6 \leq n \leq 7$ and $m \geq n-1$; $n \geq 8$ and $n-1 \leq m \leq n$ or $((q \cdot n - 2q + 1)/2 \leq m \leq (q \cdot n - q + 2)/2$ with $3 \leq q \leq n-5)$ or $m \geq (n-3)^2/2$; odd $n \geq 9$ and $((q \cdot n - 3q + 1)/2 \leq m \leq (q \cdot n - 2q)/2$ with $3 \leq q \leq (n-3)/2)$ or $((q \cdot n - q - n + 4)/2 \leq m \leq (q \cdot n - 2q)/2$ with $(n-1)/2 \leq q \leq n-5)$. We present the Ramsey numbers for 'small' paths versus fans or paths versus 'small' fans in Corollary 3.4.1, and the Ramsey numbers for paths versus 'large' fans in Corollary 3.4.2 and Corollary 3.4.3. Moreover, we give lower bounds and upper bounds for $R(P_n, F_m)$ for (odd $n \geq 11$ and $(q \cdot n - q + 4)/2 \leq m \leq (q \cdot n - 3q + n - 3)/2$ with $2 \leq q \leq (n-7)/2)$ or (even $n \geq 8$ and $(q \cdot n - q + 3)/2 \leq m \leq (q \cdot n - 2q + n - 2)/2$ with $2 \leq q \leq n-5)$ or $(n \geq 6$ and $(n+2)/2 \leq m \leq n-2)$ in Corollary 3.4.4, Corollary 3.4.5 and Corollary 3.4.6.

Corollary 3.4.1.

$$R(P_n, F_m) = \begin{cases} 1 & \text{for } n = 1 \text{ and } m \geq 2 \\ 2m+1 & \text{for } n = 2 \text{ or } n = 3 \text{ and } m \geq 2 \\ 2n-1 & \text{for } n \geq 4 \text{ and } 2 \leq m \leq (n+1)/2. \end{cases}$$

Proof. The graphs

$$\begin{cases} P_1 & \text{for } n = 1 \text{ and } m \geq 2 \\ 2mP_1 & \text{for } n = 2 \text{ and } m \geq 2 \\ mK_2 & \text{for } n = 3 \text{ and } m \geq 2 \\ 2K_{n-1} & \text{for } n \geq 4 \text{ and } 2 \leq m \leq (n+1)/2 \end{cases}$$

give the best lower bounds for $R(P_n, F_m)$ for the values of m and n in Corollary 3.4.1. Corollary 3.3.1 completes the proof. ☐

Corollary 3.4.2. *If* $(4 \leq n \leq 7$ *and* $m \geq n-1)$ *or* $(n \geq 8$ *and* $n-1 \leq m \leq n$ *or* $((q \cdot n - 2q + 1)/2 \leq m \leq (q \cdot n - q + 2)/2$ *with* $3 \leq q \leq n-5)$ *or* $m \geq (n-3)^2/2)$, *then*

$$R(P_n, F_m) = \begin{cases} 2m + n - 1 & \text{for } 2m = 1 \ mod(n-1) \\ 2m + n - 2 & \text{for other values of } m. \end{cases}$$

Proof. Let r denote the remainder of $2m$ divided by $n-1$, so $2m = p(n-1)+r$ for some $0 \leq r \leq n-2$. Then for $(4 \leq n \leq 7$ and $m \geq n-1)$ or $(n \geq 8$ and $n-1 \leq m \leq n$ or $((q \cdot n - 2q+1)/2 \leq m \leq (q \cdot n - q + 2)/2$ for $3 \leq q \leq n-5)$ or $m \geq (n-3)^2/2)$, the graphs

$$\begin{cases} (p-1)K_{n-1} \cup 2K_{n-2} & \text{for } r = 0 \\ (p+1)K_{n-1} & \text{for } r = 1 \text{ or } 2 \\ (p+r+1-n)K_{n-1} \cup (n+1-r)K_{n-2} & \text{for other values of } r \end{cases}$$

show that

$$R(P_n, F_m) > \begin{cases} 2m + n - 2 & \text{for } 2m = 1 \ mod(n-1) \\ 2m + n - 3 & \text{for other values of } m. \end{cases}$$

Corollary 3.3.3 completes the proof. ☐

Corollary 3.4.3. *If* n *is odd,* $n \geq 9$ *and either* $((q \cdot n - 3q + 1)/2 \leq m \leq (q \cdot n - 2q)/2$ *with* $3 \leq q \leq (n-3)/2)$ *or* $((q \cdot n - q - n + 4)/2 \leq m \leq (q \cdot n - 2q)/2$ *with* $(n-1)/2 \leq q \leq n-5)$, *then* $R(P_n, F_m) = 2m + n - 3$.

Proof. For odd $n \geq 9$ and $m = (q \cdot n - 2q - j)/2$ with either $(3 \leq q \leq (n-3)/2$ and $0 \leq j \leq q-1)$ or $((n-1)/2 \leq q \leq n-5$ and $0 \leq j \leq n-q-4)$, the graph $(q-j-1)K_{n-2} \cup (j+2)K_{n-3}$ shows that $R(P_n, F_m) > 2m + n - 4$. Corollary 3.3.5 completes the proof. ☐

Corollary 3.4.4. *If n is odd, $n \geq 11$ and $(q \cdot n - q + 4)/2 \leq m \leq (q \cdot n - 3q + n - 3)/2$ with $2 \leq q \leq (n-7)/2$, then*

$$2m + n - 3 \geq R(P_n, F_m) \geq$$

$$max\left\{\left\lfloor \frac{2m}{n-1} \right\rfloor (n-1) + n,\ 2m + \left\lfloor \frac{2m-1}{\lceil 2m/(n-1) \rceil} \right\rfloor\right\}.$$

Proof. Let $t = \left\lceil \frac{2m}{n-1} \right\rceil$ and let s denote the remainder of $2m-1$ divided by t. Then for m and n satisfying $\left\lfloor \frac{2m}{n-1} \right\rfloor (n-1) + n \geq 2m + \left\lfloor \frac{2m-1}{t} \right\rfloor$, the graph tK_{n-1} shows that $R(P_n, F_m) > \left\lfloor \frac{2m}{n-1} \right\rfloor (n-1) + n - 1$.

For other values of m and n, the graph $sK_{\lceil (2m-1)/t \rceil} \cup (t - s + 1)K_{\lfloor (2m-1)/t \rfloor}$ shows that $R(P_n, F_m) > 2m - 1 + \left\lfloor \frac{2m-1}{\lceil 2m/(n-1) \rceil} \right\rfloor$.

The upper bound comes from Corollary 3.3.6. □

Corollary 3.4.5. *If n is even, $n \geq 8$ and $(q \cdot n - q + 3)/2 \leq m \leq (q \cdot n - 2q + n - 2)/2$ with $2 \leq q \leq n - 5$, then*

$$2m + n - 2 \geq R(P_n, F_m) \geq$$

$$max\left\{\left\lfloor \frac{2m}{n-1} \right\rfloor (n-1) + n,\ 2m + \left\lfloor \frac{2m-1}{\lceil 2m/(n-1) \rceil} \right\rfloor\right\}.$$

Proof. Let $t = \left\lceil \frac{2m}{n-1} \right\rceil$ and let s denote the remainder of $2m-1$ divided by t. Then for m and n satisfying $\left\lfloor \frac{2m}{n-1} \right\rfloor (n-1) + n \geq 2m + \left\lfloor \frac{2m-1}{t} \right\rfloor$, the graph tK_{n-1} shows that $R(P_n, F_m) > \left\lfloor \frac{2m}{n-1} \right\rfloor (n-1) + n - 1$.

For other values of m and n, the graph $sK_{\lceil (2m-1)/t \rceil} \cup (t - s + 1)K_{\lfloor (2m-1)/t \rfloor}$ shows that $R(P_n, F_m) > 2m - 1 + \left\lfloor \frac{2m-1}{\lceil 2m/(n-1) \rceil} \right\rfloor$.

The upper bound comes from Corollary 3.3.7. □

Corollary 3.4.6. *If $n \geq 6$ and $(n+2)/2 \leq m \leq n - 2$, then*

$$2m + \left\lfloor \frac{3n}{2} \right\rfloor - 2 \geq R(P_n, F_m) \geq \begin{cases} 2n - 1 & for\ \frac{n+2}{2} \leq m \leq \frac{n + \lfloor n/3 \rfloor}{2} \\ 3m - 1 & for\ \frac{n + \lfloor n/3 \rfloor}{2} < m \leq n - 2. \end{cases}$$

Proof. For $n \geq 6$ and $\frac{n+2}{2} \leq m \leq \frac{n+\lfloor n/3 \rfloor}{2}$, the graph $2K_{n-1}$ shows that $R(P_n, F_m) > 2n-2$. For $n \geq 6$ and $\frac{n+\lfloor n/3 \rfloor}{2} < m \leq n-2$, the graph $K_m \cup 2K_{m-1}$ shows that $R(P_n, F_m) > 3m-2$.

The upper bound comes from Theorem 3.3.8. □

Chapter 4

λ-Backbone Colorings

Abstract In this chapter we study combinatorial and algorithmic aspects of λ-backbone coloring of graphs where the backbone is a collection of pairwise disjoint stars or a perfect matching. We determine a relation between the λ-backbone coloring numbers and the chromatic numbers. We also study the special case where the graph is a planar graph and the backbone is a perfect matching. Besides that, we study the λ-backbone coloring numbers of split graphs with star backbones or matching backbones or tree backbones. Finally, we study the computational complexity of computing the λ-backbone coloring number of a graph with a star backbone or a matching backbone or a tree backbone or a path backbone.

4.1 Introduction

Let $H = (V, E_H)$ be a spanning subgraph of $G = (V, E)$ and let $\lambda \geq 2$. For convenience we repeat some definitions. Let $G = (V, E)$ be a graph. A *vertex coloring* $f : V \to \{1, 2, 3, \ldots\}$ of V is *proper*, if $|f(u) - f(v)| \geq 1$ holds for all edges $uv \in E$. A proper vertex coloring $f : V \to \{1, \ldots, k\}$ is called a *k-coloring*, and the *chromatic number* $\chi(G)$ is the smallest integer k for which there exists a k-coloring. A vertex coloring f is a λ-*backbone coloring* of (G, H), if it is proper and if additionally $|f(u) - f(v)| \geq \lambda$ holds for all edges $uv \in E_H$.

The λ-*backbone coloring number* $\text{BBC}_\lambda(G,H)$ of (G,H) is the smallest integer ℓ for which there exists a λ-backbone coloring $f : V \to \{1,\dots,\ell\}$. A spanning subgraph H of a graph G is called a *star backbone*, a *matching backbone*, a *tree backbone* or a *path backbone* of G if H is a collection of pairwise disjoint stars, a perfect matching, a tree or a path, respectively.

We present a relation between the λ-backbone coloring number and the chromatic number where the backbone is a star backbone or a matching backbone in Section 4.2 and Section 4.3, respectively. In Section 4.4 we consider planar graphs with matching backbones. In Subsection 4.5.1, Subsection 4.5.2 and Subsection 4.5.3 we present sharp upper bounds for the λ-backbone coloring numbers of split graphs with star backbones, matching backbones or tree backbones, respectively. Finally, in Subsection 4.6.1 and in Subsection 4.6.2 we present the computational complexity of computing the λ-backbone coloring number where the backbone is a collection of pairwise disjoint stars or a perfect matching, and where the backbone is a tree or a path, respectively.

4.2 λ-Backbone coloring numbers of graphs with star backbones

In [46] we showed for star backbones S of G the number of colors needed for a λ-backbone coloring of (G,S) can roughly differ by a multiplicative factor of at most $2-\frac{1}{\lambda}$ from the chromatic number $\chi(G)$. Their precise behavior is summarized in the following theorem.

Theorem 4.2.1. *For $\lambda \geq 2$ the function $\mathcal{S}_\lambda(k)$ takes the following values:*

(a) $\mathcal{S}_\lambda(2) = \lambda + 1$;

(b) for $3 \leq k \leq 2\lambda - 3$*:* $\mathcal{S}_\lambda(k) = \lceil \frac{3k}{2} \rceil + \lambda - 2$;

(c) for $2\lambda - 2 \leq k \leq 2\lambda - 1$ *with* $\lambda \geq 3$*:* $\mathcal{S}_\lambda(k) = k + 2\lambda - 2$; $\mathcal{S}_2(3) = 5$;

(d) for $k = 2\lambda$ *with* $\lambda \geq 3$*:* $\mathcal{S}_\lambda(k) = 2k - 1$; $\mathcal{S}_2(4) = 6$;

(e) for $k \geq 2\lambda + 1$*:* $\mathcal{S}_\lambda(k) = 2k - \lfloor \frac{k}{\lambda} \rfloor$.

Proof. We divide the proof into two parts as follows.

Part 1 Proof of the upper bounds.
If $k = 2$ then G is bipartite, and we use colors 1 and $\lambda + 1$. For $k \geq 3$, let $G = (V, E)$ be a graph with $\chi(G) = k$ and let $V_1, \ldots, V_k$ denote the corresponding independent sets in a k-coloring. Let $S = (V, E_S)$ be a star backbone of G.

First, we will give upper bounds for $\mathcal{S}_\lambda(k)$ in case $3 \leq k \leq 2\lambda - 3$. Consider the following color sets:

- $C_i = \{i, k + \lambda - 1 - i\}$ for $i = 1, \ldots, \lfloor \frac{k}{2} \rfloor$;
- $C_i = \{i, 2k + \lambda - 1 - i\}$ for $i = \lfloor \frac{k}{2} \rfloor + 1, \ldots, k$.

The union of these k color sets consists of $2k$ colors, namely the colors in $\{1, \ldots, k\}$ together with the colors in $\{k+\lambda-1-\lfloor \frac{k}{2} \rfloor, \ldots, 2k+\lambda-1-(\lfloor \frac{k}{2} \rfloor+1)\}$. The largest color used is $2k + \lambda - 1 - (\lfloor \frac{k}{2} \rfloor + 1) = \lceil \frac{3k}{2} \rceil + \lambda - 2$.

We construct a λ-backbone coloring of (G, S) such that every vertex in V_i $(i = 1, \ldots, k)$ is colored with a color in C_i.

For $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$ a root vertex in V_i is colored with the first color of C_i. Its leaves in a set V_j are colored with the second color of C_j. This does not give any conflict, since the smallest gap appears if the root vertex is in $V_{\lfloor \frac{k}{2} \rfloor}$ and one of its leaves is in $V_{\lfloor \frac{k}{2} \rfloor - 1}$, or the other way around. In both cases this gap is $k + \lambda - 1 - \lfloor \frac{k}{2} \rfloor - (\lfloor \frac{k}{2} \rfloor - 1) = k + \lambda - 2\lfloor \frac{k}{2} \rfloor \geq \lambda$.

For $\lfloor \frac{k}{2} \rfloor + 1 \leq i \leq k$ a root vertex in V_i is colored with the second color of C_i. Its leaves in a set V_j are colored with the first color of C_j. This is possible, since the smallest gap appears if the root vertex is in V_k and one of its leaves is in V_{k-1}, or the other way around. In both cases this gap is $2k + \lambda - 1 - k - (k - 1) = \lambda$.

For the case $2\lambda - 2 \leq k \leq 2\lambda - 1$ with $\lambda \geq 3$, and the case $k = 2\lambda - 1$ with $\lambda = 2$ we use color sets:

- $C_i = \{i, 2\lambda - 1 + i\}$ for $i = 1, \ldots, k - 1$;
- $C_k = \{k\}$.

Note that these k color sets are pairwise disjoint. The union of these sets consists of all the colors in $\{1, \ldots, k\}$ together with all the colors in $\{2\lambda, \ldots, 2\lambda + k - 2\}$.

We construct a λ-backbone coloring of (G, S) such that for $1 \leq i \leq k$ every vertex in V_i is colored with a color in C_i. This means that vertices in V_k are assigned color k. A root vertex in V_i for $1 \leq i \leq \lceil \frac{k}{2} \rceil - 1$ is assigned color i. A root vertex in V_i for $\lceil \frac{k}{2} \rceil \leq i \leq k-1$ is assigned color $2\lambda - 1 + i$. This way the distance between the color of a root vertex not in V_k and the color k of a vertex in V_k is at least λ.

All other vertices in V are colored greedily and in arbitrary order: Let $v \in V_i$ $(1 \leq i \leq k-1)$ be a leaf vertex of a star S with root w. Let x be the color assigned to w. Then colors $x - \lambda + 1, \ldots, x + \lambda - 1$ are forbidden colors for v. The distance between $x + \lambda - 1$ and $x - \lambda + 1$ is $2\lambda - 2$. Since the two colors in C_i have pairwise distance $2\lambda - 1$, at least one of them is feasible for v.

For the case $k = 2\lambda$ with $\lambda \geq 3$ we use color sets:

- $C_i = \{i, 2\lambda - 1 + i\}$ for $i = 1, \ldots, k-1$;
- $C_k = \{4\lambda - 1\}$.

By similar arguments as in the previous case we can construct a λ-backbone coloring using at most $2k - 1$ colors.

For proving that $\mathcal{S}_2(4) \leq 6$ we use color sets $C_1 = \{1\}$, $C_2 = \{3, 2\}$, $C_3 = \{4, 5\}$ and $C_4 = \{6\}$, and choose the first colors in the sets for the root vertices.

For the case $k \geq 2\lambda + 1$ we use color sets:

- $C_i = \{(i-1)\lambda + 1\}$ for $i = 1, \ldots, \lfloor \frac{k}{\lambda} \rfloor$;
- $C'_i = \{\lceil \frac{i \cdot \lambda}{\lambda - 1} \rceil, k + i\}$ for $i = 1, \ldots, \lfloor \frac{k}{\lambda} \rfloor (\lambda - 1)$;
- $C''_i = \{\lfloor \frac{k}{\lambda} \rfloor \lambda + i, k + \lfloor \frac{k}{\lambda} \rfloor (\lambda - 1) + i\}$ for $i = 1, \ldots, k - \lfloor \frac{k}{\lambda} \rfloor \lambda$ and $k > \lfloor \frac{k}{\lambda} \rfloor \lambda$.

If $j = s(\lambda - 1) + t$ for some integers $s \geq 0$ and $0 \leq t \leq \lambda - 2$, then $\lceil \frac{j \cdot \lambda}{\lambda - 1} \rceil$ is equal to $s \cdot \lambda$ in case $t = 0$ and to $s \cdot \lambda + t + 1$ in case $t > 0$. Then $C_i \cap C'_j$ is empty for all $1 \leq i \leq \lfloor \frac{k}{\lambda} \rfloor$ and $1 \leq j \leq \lfloor \frac{k}{\lambda} \rfloor (\lambda - 1)$. Hence the k color sets as defined above are pairwise disjoint, and cover the whole range $1, \ldots, 2k - \lfloor \frac{k}{\lambda} \rfloor$.

We construct a λ-backbone coloring of (G, S) as follows. For $1 \leq i \leq \lfloor \frac{k}{\lambda} \rfloor$ vertices in V_i are assigned the color in C_i. Note that these colors are at least λ apart from each other. For $1 \leq i \leq \lfloor \frac{k}{\lambda} \rfloor (\lambda - 1)$ a root vertex in $V_{\lfloor \frac{k}{\lambda} \rfloor + i}$ is assigned the second color in C'_i. For $1 \le i \le k - \lfloor \frac{k}{\lambda} \rfloor \lambda$ a root vertex in $V_{\lfloor \frac{k}{\lambda} \rfloor(\lambda)+i}$ is assigned the second color in C''_i. So far we have not created any conflict, since a second color in a set C''_i is larger than a second color in any set C'_j, and the smallest gap between a second color in a set C'_j and a color in a set C_h is $k+1-((\lfloor \frac{k}{\lambda} \rfloor - 1)\lambda + 1) = k - \lfloor \frac{k}{\lambda} \rfloor \lambda + \lambda \ge \lambda$.

Note that both the distance $k + i - \lceil \frac{i \cdot \lambda}{\lambda - 1} \rceil$ between two colors in color set C'_i and the distance $k + \lfloor \frac{k}{\lambda} \rfloor (\lambda - 1) + j - (\lambda \lfloor \frac{k}{\lambda} \rfloor + j)$ between two colors in color set C''_j are at least

$$k + \lfloor \tfrac{k}{\lambda} \rfloor (\lambda - 1) - \lfloor \tfrac{k}{\lambda} \rfloor \lambda = k - \lfloor \tfrac{k}{\lambda} \rfloor = \lceil \tfrac{k(\lambda-1)}{\lambda} \rceil \ge \lceil \tfrac{(2\lambda+1)(\lambda-1)}{\lambda} \rceil = 2\lambda - 1.$$

This means that just as in previous cases all other vertices in V can be colored greedily and in arbitrary order.

Part 2 Proof of the lower bounds.
Let $\lambda \ge 2$. The case $k = 2$ is trivial. For $k \ge 3$, we consider a complete k-partite graph G that consists of k independent sets $V_1, \ldots, V_k$ that are all of cardinality k. Let $S = (V, E_S)$ be a star backbone of G that consists of k stars S_{k-1}. Each V_i contains exactly one root vertex of some star in S and its other $k-1$ vertices are leaves from $k-1$ different stars.

Consider some fixed λ-backbone ℓ-coloring of (G, S). Since G is complete k-partite, any color that shows up in some set V_i can not show up in any V_j with $j \ne i$. We denote by C_i the set of colors that are used on vertices in V_i. If $|C_i| = 1$, then V_i is called *monochromatic*, and if $|C_i| \ge 2$, then V_i is called *polychromatic*. We denote by s_1 and s_2 the number of monochromatic and polychromatic sets, respectively. Then we immediately have $s_1 + s_2 = k$ and $s_1 + 2s_2 \le \ell$ implying

$$s_1 \ge 2k - \ell. \tag{4.1}$$

A root in a monochromatic set is called *monochromatic* as well. A *root color* is a color that is used for a root. From the above it is clear that each root has a different color. So we have the following simple observation.

Observation 4.2.2. *The number of different root colors is equal to k.*

Since all stars in S have (exactly) one leaf in any set that does not contain their root vertex, we immediately have the following.

Observation 4.2.3. *If x is a root color, then there are at least $k-1$ other colors that have distance at least λ to x.*

Observation 4.2.4. *If x is the color for the root in set V_i and V_j $(j \neq i)$ is a monochromatic set colored by y, then the distance between x and y is at least λ.*

If $s_2 = 0$, then $s_1 = k$, and by Observation 4.2.4 there are at least $(k-1)$ gaps of at least $\lambda - 1$ colors that can not be used to color the k roots. Then the total number of colors needed is at least $(k-1)(\lambda-1)+k = (k-1)\lambda+1$. If $s_2 > 0$, the same observation implies that there are at least s_1 gaps of at least $\lambda - 1$ colors. In this case the total range of colors is at least $s_1(\lambda-1)+k$. This way we have found

$$\ell \geq \begin{cases} (k-1)\lambda+1 & \text{if } s_2 = 0; \\ s_1(\lambda-1)+k & \text{if } s_2 > 0. \end{cases} \tag{4.2}$$

Due to Observation 4.2.3 we can prove the following lemma.

Lemma 4.2.5. *If $\ell \leq k+2\lambda-3$ then only colors from $A = \{1,\dots,\ell-k-\lambda+2\}$ and $B = \{k+\lambda-1,\dots,\ell\}$ can be assigned to root vertices.*

Proof. Suppose that a root v is assigned color c with c in $\{\ell-k-\lambda+3,\dots,k+\lambda-2\}$. By Observation 4.2.3 there have to be at least $k-1$ colors with distance at least λ from c. If $\lambda+1 \leq c \leq \ell-\lambda$, only colors in $\{1,\dots,c-\lambda\}$ and in $\{c+\lambda,\dots,\ell\}$ can be used. These sets together contain $c-\lambda+\ell-(c+\lambda)+1 = \ell-2\lambda+1 \leq k-2$ colors. Hence either $c \leq \lambda$ or $c \geq \ell-\lambda+1$ holds. If $c \leq \lambda$, then only colors in $\{c+\lambda,\dots,\ell\}$ are at distance at least λ. The cardinality of this set is $\ell-(c+\lambda)+1 \leq \ell-(\ell-k-\lambda+3)-\lambda+1 = k-2$. If $c \geq \ell-\lambda+1$, then only colors in $\{1,\dots,c-\lambda\}$ are at distance at least λ. The cardinality of this set is $c-\lambda \leq k+\lambda-2-\lambda = k-2$. $\square$

First we consider the case $3 \leq k \leq 2\lambda-3$. Suppose that there exists a λ-backbone coloring of (G,S) with $\ell = \lceil \frac{3k}{2} \rceil + \lambda - 3$ colors. Then $\ell \leq k+2\lambda-3$ and by Lemma 4.2.5 only colors in $A = \{1,\dots,\lceil \frac{k}{2} \rceil - 1\}$ and colors in $B = \{k+\lambda-1,\dots,\lceil \frac{3k}{2} \rceil + \lambda - 3\}$ can be used on roots. Since the total number of colors in A united with B is $2(\lceil \frac{k}{2} \rceil - 1) < k$, we obtain a contradiction by Observation 4.2.2.

Let $2\lambda-2 \leq k \leq 2\lambda-1$ with $\lambda \geq 3$, or $2\lambda-1 \leq k \leq 2\lambda$ with $\lambda = 2$. Suppose that there exists a λ-backbone coloring of (G,S) with $\ell = k+2\lambda-3$ colors. By Lemma 4.2.5 only colors in $A = \{1,\dots,\lambda-1\}$ and $B = \{k+\lambda-1,\dots,k+2\lambda-3\}$ may be used on roots. By (4.1), there exists at least one monochromatic set.

Let y be the (root) color used on this set. Without loss of generality we may assume that y is in A. By Observation 4.2.4 all other $k-1$ root colors must be in B. However, B contains $\lambda - 1 < k - 1$ colors.

Let $k = 2\lambda$ with $\lambda \geq 3$. Suppose that there exists a λ-backbone coloring of (G, S) with $2k - 2$ colors. If $s_2 = 0$, then by (4.2) we have $\ell \geq (k-1)\lambda + 1 \geq 3k - 2$. Hence $s_2 > 0$. By (4.1), $s_1 \geq 2$. Together with (4.2) this implies that $s_1 = 2$, and $\ell = s_1(\lambda - 1) + k$. Then there are only three feasible ways to choose k different root colors:

- monochromatic roots: $1, \lambda + 1$, other roots: $2\lambda + 1, \ldots, 4\lambda - 2$;
- monochromatic roots: $1, 4\lambda - 2$, other roots: $\lambda + 1, \ldots, 3\lambda - 2$;
- monochromatic roots: $3\lambda - 2, 4\lambda - 2$, other roots: $1, \ldots, 2\lambda - 2$.

Consider the first case. Since color $2\lambda + 1$ is a root color, in every other color set there must be at least one color that has distance at least λ to color $2\lambda+1$. This condition is already met for the sets with root color 1, root color $\lambda+1$ or root colors $3\lambda+1, \ldots, 4\lambda-2$. However, the sets with root colors $2\lambda+2, \ldots, 3\lambda$ need an extra color. Hence, we need $\lambda - 1$ extra colors that have distance at least λ to color $2\lambda + 1$. There are exactly $\lambda - 1$ such colors available, namely colors $2, \ldots, \lambda$. So one of the colors $2, \ldots, \lambda$ must be in the same set with color $2\lambda + 2$.

Simultaneously, since color $2\lambda + 2$ is also a root color, in every other color set there must be at least one color that has distance at least λ to color $2\lambda + 2$. This condition is not met yet for the sets with root color $2\lambda + 1$ or root colors $2\lambda + 3, \ldots, 3\lambda + 1$. To satisfy the condition, we need λ extra colors that have distance at least λ to color $2\lambda+2$. The only available colors are colors $2, \ldots, \lambda$ and color $\lambda + 2$. This implies that none of the colors $2, \ldots, \lambda$ can be in the same set with color $2\lambda + 2$, which is a contradiction.

The other two cases can be proven by the same argument.

Let $k \geq 2\lambda + 1$. Suppose that there exists a λ-backbone coloring of (G, S) with $\ell = 2k - \lfloor \frac{k}{\lambda} \rfloor - 1$ colors. Suppose that $s_2 = 0$. Then there are only monochromatic sets, i.e., $s_1 = k$. By (4.2) the total number of colors needed is at least $(k-1)\lambda + 1$. However, the difference between this number and ℓ is

$$(k-1)\lambda + 1 - (2k - \lfloor \tfrac{k}{\lambda} \rfloor - 1) = k(\lambda - 2) + \lfloor \tfrac{k}{\lambda} \rfloor - \lambda + 2 \geq 2\lambda^2 - 4\lambda + 2 > 0.$$

Suppose that $s_2 > 0$. Write $k = a\lambda + r$ for some integers $a \geq 2$ and $0 \leq r \leq \lambda - 1$. By (4.1), $s_1 \geq \lfloor \frac{k}{\lambda} \rfloor + 1$ holds. Together with (4.2) this implies that we need at least $(\lfloor \frac{k}{\lambda} \rfloor + 1)(\lambda - 1) + k$ colors. However, the difference between this number and ℓ is $(\lfloor \frac{k}{\lambda} \rfloor + 1)(\lambda - 1) + k - (2k - \lfloor \frac{k}{\lambda} \rfloor - 1) = \lfloor \frac{k}{\lambda} \rfloor \lambda + \lambda - k = \lambda - r > 0$. □

4.3 λ-Backbone coloring numbers of graphs with matching backbones

In [46] we studied the case where $\lambda \geq 2$ and the backbone is a perfect matching. We determine all values $\mathcal{M}_\lambda(k)$ and observe that they roughly grow like $(2 - \frac{2}{\lambda+1})k$. Their precise behavior is summarized in the following theorem.

Theorem 4.3.1. *For $k \geq 2$ the function $\mathcal{M}_\lambda(k)$ takes the following values:*

(a) for $2 \leq k \leq \lambda$: $\mathcal{M}_\lambda(k) = \lambda + k - 1$;

(b) for $\lambda + 1 \leq k \leq 2\lambda$: $\mathcal{M}_\lambda(k) = 2k - 2$;

(c) for $k = 2\lambda + 1$: $\mathcal{M}_\lambda(k) = 2k - 3$;

(d) for $k = t(\lambda + 1)$ with $t \geq 2$: $\mathcal{M}_\lambda(k) = 2\lambda \cdot t$;

(e) for $k = t(\lambda + 1) + c$ with $t \geq 2$, $1 \leq c < \frac{\lambda+3}{2}$: $\mathcal{M}_\lambda(k) = 2\lambda \cdot t + 2c - 1$;

(f) for $k = t(\lambda + 1) + c$ with $t \geq 2$, $\frac{\lambda+3}{2} \leq c \leq \lambda$: $\mathcal{M}_\lambda(k) = 2\lambda \cdot t + 2c - 2$.

Proof. We divide the proof into two parts as follows.

Part 1 Proof of the upper bounds.
If $k = 2$ then G is bipartite, and we use colors 1 and $\lambda + 1$. For $k \geq 3$, let $G = (V, E)$ be a graph with $\chi(G) = k$ and let $V_1, \ldots, V_k$ denote the corresponding independent sets in a k-coloring. Let $M = (V, E_M)$ be a matching backbone of G. For a vertex v in G, we denote by $mn(v)$ the only neighbor of v in M.

First, we will give upper bounds for $\mathcal{M}_\lambda(k)$ in case $3 \leq k \leq \lambda$. Consider the following color sets:

- $C_1 = \{1\}$
- $C_j = \{j, \lambda + j - 1\}$ for $j = 2, \ldots, k-1$;
- $C_k = \{\lambda + k - 1\}$.

Note that these k color sets are pairwise disjoint. The union of these sets consists of all the colors in $\{1, \ldots, k-1\}$ together with all the colors in $\{\lambda + 1, \ldots, \lambda + k - 1\}$. Moreover, the color of set C_1 and the color of set C_k are at distance $\lambda + k - 2 \geq \lambda$. For $2 \leq j \leq k-1$ we have that the color of set C_1 and the second color of set C_j are at distance $\lambda + j - 2 \geq \lambda$, and that the first color of set C_j and the color of set C_k are at distance $\lambda + k - 1 - j \geq \lambda$. For $2 \leq m < n \leq k-1$, the first color of the set C_m and the second color of the set C_n are at distance at least λ.

These properties enable us to construct a λ-backbone coloring of (G, M) such that each set V_i gets a color from set C_i. Then vertices in V_1 get color 1 and vertices in V_k get color $\lambda + k - 1$. The choice for all other vertices depends on the incidences with edges from M. Let $v \in V_j$ $(j = 2, \ldots, k-1)$ and $uv \in E_M$.

- if $u \in V_1$: $b(v) = \lambda + j - 1$;
- if $u \in V_k$: $b(v) = j$;
- if $u \in V_m$ $(1 < m < j)$: $b(v) = \lambda + j - 1$;
- if $u \in V_n$ $(j < n < k)$: $b(v) = j$.

For the case $\lambda + 1 \leq k \leq 2\lambda$ we use color sets:

- $C_1 = \{1\}$;
- $C_j = \{j, k + j - 2\}$ for $j = 2, \ldots, k-1$;
- $C_k = \{2k - 2\}$.

By the same arguments as in the first case we can construct a λ-backbone coloring using at most $2k - 2$ colors.

For the case $k = 2\lambda + 1$ we use color sets:

- $C_i = \{i \cdot \lambda + 1\}$ for $i = 0, \ldots, 3$;

- $C_{1,j} = \{j, 2\lambda + j\}$ for $j = 2, \ldots, \lambda$;
- $C_{2,j} = \{\lambda + j, 3\lambda + j\}$ for $j = 2, \ldots, \lambda - 1$ and $\lambda \geq 3$.

These k color sets are pairwise disjoint. The union of these sets is equal to $\{1, \ldots, 4\lambda - 1\} \backslash \{2\lambda\}$.

We construct a λ-backbone coloring of (G, M) that for $i = 0, \ldots, 3$ assigns the color of set C_i to the vertices in V_{i+1}. This does not give any conflict, since the colors of the sets C_i have distance at least λ to each other. For $j = 2, \ldots, \lambda$ vertices in V_{j+3} are assigned a suitable color from set $C_{1,j}$, and for $j = 2, \ldots \lambda - 1$ vertices in $V_{\lambda+j+2}$ are assigned a suitable color from set $C_{2,j}$. Since colors within a set $C_{1,j}$ and within a set $C_{2,j}$ are at distance 2λ, we can color the vertices in V_{j+3} $(j = 2, \ldots, 2\lambda - 2)$ greedily and in arbitrary order (cf. the proof of the upper bounds in Theorem 4.2.1).

The remaining cases follow by simple modifications of arguments that have been used for case $k = 2\lambda + 1$.

For $k = t(\lambda + 1)$ with $t \geq 2$ we use color sets:

- $C_i = \{i \cdot \lambda + 1\}$ for $i = 0, \ldots, 2t - 1$;
- $C_{i,j} = \{i \cdot \lambda + j, (t + i)\lambda + j\}$ for $i = 0, \ldots, t - 1$ and $j = 2, \ldots, \lambda$.

For $k = t(\lambda + 1) + c$ with $t \geq 2$, $1 \leq c < \frac{\lambda+3}{2}$ we use color sets:

- $C_i = \{i \cdot \lambda + 1\}$ for $i = 0, \ldots, 2t$;
- $C_{0,j} = \{j, 2t \cdot \lambda + 2j - 2\}$ for $j = 2, \ldots, c$ and $c \geq 2$;
- $C_{0,j} = \{j, t \cdot \lambda + j\}$ for $j = c + 1, \ldots, \lambda$ and $c < \lambda$;
- $C_{i,j} = \{i \cdot \lambda + j, (t + i)\lambda + j\}$ for $i = 1, \ldots, t - 1$ and $j = 2, \ldots, \lambda$;
- $C_{t,j} = \{t \cdot \lambda + j, 2t \cdot \lambda + 2j - 1\}$ for $j = 2, \ldots, c$ and $c \geq 2$.

For $k = t(\lambda + 1) + c$ with $t \geq 2$, $\frac{\lambda+3}{2} \leq c \leq \lambda$ we use color sets:

- $C_i = \{i \cdot \lambda + 1\}$ for $i = 0, \ldots, 2t$;
- $C_{2t+1} = \{2t \cdot \lambda + 2c - 2\}$;

- $C_{0,j} = \{j, 2t \cdot \lambda + 2j - 2\}$ for $j = 2, \ldots, c-1$;
- $C_{0,j} = \{j, t \cdot \lambda + j\}$ for $j = c, \ldots, \lambda$;
- $C_{i,j} = \{i \cdot \lambda + j, (t+i)\lambda + j\}$ for $i = 1, \ldots, t-1$ and $j = 2, \ldots, \lambda$;
- $C_{t,j} = \{t \cdot \lambda + j, 2t \cdot \lambda + 2j - 1\}$ for $j = 2, \ldots, c-1$.

Part 2 Proof of the lower bounds.
Let $\lambda \geq 2$. The case $k = 2$ is trivial. For $k \geq 3$, we consider a complete k-partite graph G that consists of k independent sets $V_1, \ldots, V_k$ that are all of cardinality $k-1$. For $1 \leq i \leq k$, let $\{v_{i,j} \mid 1 \leq j \leq k,\ j \neq i\}$ be the vertices of V_i, and let M be a matching backbone of G such that $E_M = \{v_{i,j}v_{j,i} \mid 1 \leq i < j \leq k\}$.

Consider some fixed λ-backbone ℓ-coloring of (G, M). Since G is complete k-partite, any color that shows up in some set V_i can not show up in any V_j with $j \neq i$. Again we denote by C_i the set of colors that are used on vertices in V_i. Recall that a set V_i is called *monochromatic* if $|C_i| = 1$, and *polychromatic* if $|C_i| \geq 2$. Again we denote by s_1 and s_2 the number of monochromatic and polychromatic sets, respectively. Let $m \leq \ell$ be the number of different colors used on V. Then we immediately have $s_1 + s_2 = k$ and $s_1 + 2s_2 \leq m$ implying

$$s_1 \geq 2k - m. \tag{4.3}$$

Since there exists a matching edge between any two independent sets V_i and V_j, we obtain the following observations.

Observation 4.3.2. *If x is a color used on a monochromatic set, then there are at least $k-1$ other colors that have distance at least λ to x.*

Observation 4.3.3. *If color x is assigned to monochromatic set V_i, and color y is assigned to monochromatic set V_j, then the distance between x and y is at least λ.*

The last observation yields $\ell \geq \lambda(s_1 - 1) + 1$. Together with (4.3) and $m \leq \ell$ this implies that

$$\ell \geq \frac{2\lambda \cdot k}{\lambda + 1} - \frac{\lambda - 1}{\lambda + 1}. \tag{4.4}$$

Using a similar argumentation as in the proof of Lemma 4.2.5 we can prove the following lemma. We omit the details.

Lemma 4.3.4. *If $\ell \leq k+2\lambda-3$ then only colors from $A = \{1,\ldots,\ell-k-\lambda+2\}$ and $B = \{k+\lambda-1,\ldots,\ell\}$ can be assigned to monochromatic sets.*

We will prove the lower bounds.

In case $k = t(\lambda+1)$ with $t \geq 1$ inequality (4.4) yields $\ell \geq 2t\cdot\lambda - \frac{\lambda-1}{\lambda+1} = 2t\cdot\lambda - 1 + \frac{2}{\lambda+1}$. Since ℓ is an integer, this implies $\ell \geq 2t\cdot\lambda$. The cases $k = t(\lambda+1)+c$ with $t \geq 2$ and $1 \leq c \leq \lambda$ follow by the same argument.

Let $3 \leq k \leq \lambda$. Suppose that (G, M) has a λ-backbone coloring with $\ell = \lambda+k-2$ colors. By Lemma 4.3.4, $s_1 = 0$ holds. Colors $k-1,\ldots,\lambda$ can not be used at all, since there is no color in $\{1,\ldots,\lambda+k-2\}$ that has distance at least λ to one of them. So we can only use colors in $\{1,\ldots,k-2\}$ and $\{\lambda+1,\ldots,\lambda+k-2\}$. Then the total number m of different colors is at most $2(k-2)$. Hence, by (4.3) we find that $s_1 > 0$.

Let $\lambda+2 \leq k \leq 2\lambda$. Suppose that (G, M) has a λ-backbone coloring with $\ell = 2k-3$ colors. By (4.3), $s_1 \geq 3$ holds. By Lemma 4.3.4, only monochromatic colors in $A = \{1,\ldots,k-\lambda-1\}$ and $B = \{k+\lambda-1,\ldots,2k-3\}$ can be used. Both sets have $k-\lambda-1 \leq \lambda-1$ elements. Then by Observation 4.3.3 at most one color in A and at most one color in B can be used for monochromatic sets. Hence we find $s_1 \leq 2$.

The case $k = 2\lambda+1$ can be proven analogously to the previous case. □

4.4 λ-Backbone coloring numbers of planar graphs with matching backbones

There are many open problems about backbone colorings. We refer to [7] for details. In this section we only focus on some open problems for planar graphs. The Four-Color Theorem together with Theorem 4.3.1 implies that $\text{BBC}_2(G, M) \leq 6$ holds for any planar graph G with a matching backbone M. It seems likely that this bound 6 is not best possible. However, the planar graph G_1 with the indicated matching backbone M consisting of edges ab', bc', cd', da' as in Figure 4.1 shows that one can not improve this bound to 4.

We prove here that we can not find a backbone coloring of (G_1, M) with color set $\{1, 2, 3, 4\}$. First of all observe that G_1 can be obtained from a plane embedding of the K_4 induced by the vertices a, b, c, d, by putting a new vertex in each face and adding edges from this new vertex to the three vertices on the boundary of the face, and assigning the label x' to the new vertex in the triangular face bounded by the cycle $uvwu$, where $\{u, v, w, x\} = \{a, b, c, d\}$. Suppose that we only use colors $1, 2, 3, 4$, it is clear from this construction that a, b, c and d get different colors, and that the colors of a vertex and its primed counterpart are the same. Without loss of generality assume that a and a' get color 2. Then both b' and d must get color 4, a contradiction. It is routine to check that $\text{BBC}_2(G_1, M) = 5$.

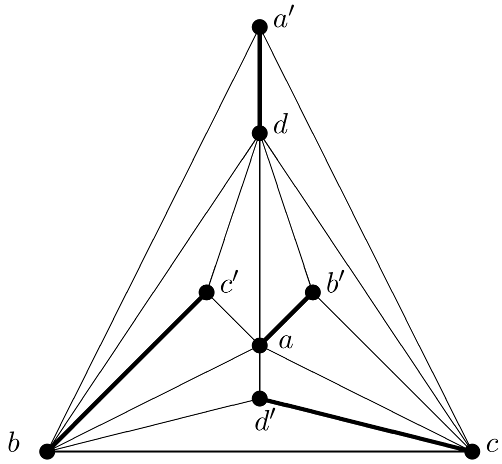


Figure 4.1: A graph G_1 with a matching backbone M such that $\text{BBC}(G_1, M) = 5$.

The following problems are still open.

Problem 4.4.1. *Is* $\text{BBC}_2(G, M) \leq 5$ *for any planar graph* G *with a matching backbone* M*?*

Problem 4.4.2. *How to prove* $\text{BBC}_2(G, M) \leq 6$ *without using the Four-Color Theorem?*

Now we recall a special kind of 2-backbone coloring, and prove a sharp result with respect to the upper bound on the number of colors needed to color planar graphs. Let $H = (V, E_H)$ be a backbone of graph $G = (V, E_G)$. A 2-backbone coloring $f : V \to \{1, \dots, \ell\}$ of (G, H) is called an *ℓ-cyclic 2-backbone coloring* of (G, H), if no edge in E_H connects two vertices with color 1 and color ℓ in V. In a 2-backbone coloring we say that two colors x and y are *adjacent* if $|x - y| \leq 1$. In an ℓ-cyclic 2-backbone coloring we also say that color 1 and color ℓ are adjacent.

For the proof of Theorem 4.4.4 below we first construct the following useful gadget.

Lemma 4.4.3. *Let H be a graph with the matching M consisting of edges ab, cd, eu, fg and hi as in Figure 4.2(a). Let G be a graph with a matching backbone M'. If $H \subset G$ and $M \subset M'$, then vertex u and vertex v can not be colored with two adjacent colors in a 5-cyclic 2-backbone coloring of (G, M').*

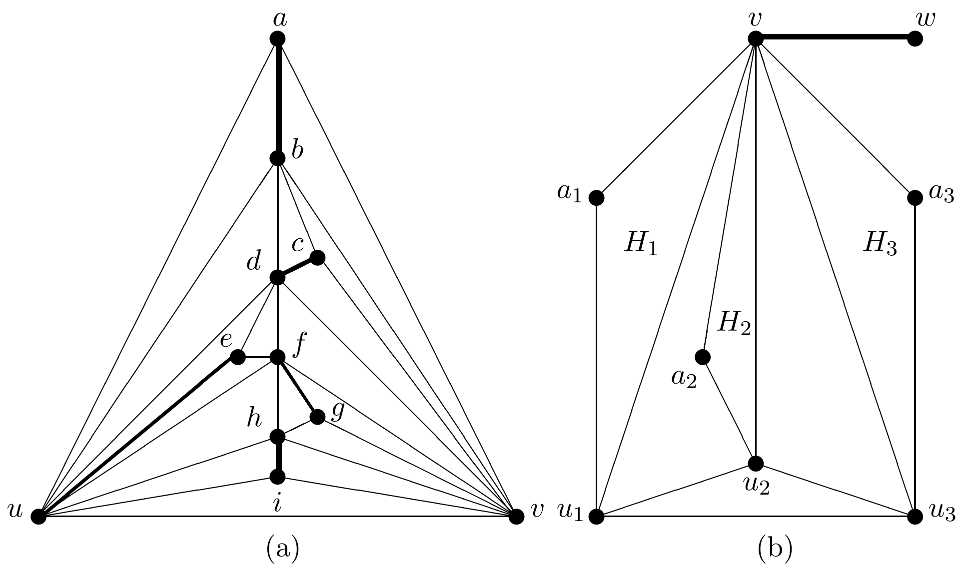


Figure 4.2: (a) A graph H with matching M (b) A planar graph G_2

Proof. Suppose that vertex u and vertex v can be colored with two adjacent colors in a 5-cyclic 2-backbone coloring of (G, M'). Since we use a 5-cyclic 2-backbone coloring, we can without loss of generality assume that vertex u is colored with color 1 and vertex v is colored with color 2. This leaves us with three possible colors for vertex d, i.e., color 3, color 4 or color 5.

- If vertex d is colored with color 3, then vertex e must get color 4. Continuing this way, vertex f gets color 5, vertex g gets color 3 and vertex h gets color 4. Since there is no feasible color for vertex i, this implies a contradiction.

- If vertex d is colored with color 4, then vertex e gets color 3, vertex f gets color 5, vertex g gets color 3 and vertex h gets color 4. Again, we find a contradiction, since there is no feasible color for vertex i.

- If vertex d is colored with color 5, then vertex c must get color 3 and the only feasible color for vertex b is color 4. We get a contradiction, since there is no feasible color for vertex a. This completes the proof of Lemma 4.4.3.

□

Theorem 4.4.4.

(a) Let G be a planar graph with a matching backbone M. Then (G, M) has a 6-cyclic 2-backbone coloring.

(b) There exist planar graphs that do not have a 5-cyclic 2-backbone coloring where the backbone is a perfect matching.

Proof. (a) By the Four-Color Theorem, we obtain that the chromatic number of a planar graph G is at most 4. We can construct a 6-cyclic 2-backbone coloring b of (G, M) by replacing the colors of a 4-coloring c of G as follows:

- if $c(v) = 1$: $b(v) = 1$;

- if $c(v) = 2$: $b(v) = 3$;

- if $c(v) = 3$: $b(v) = 5$;

- if $c(v) = 4$ and $c(mn(v)) = 1$: $b(v) = 4$;

- if $c(v) = 4$ and $c(mn(v)) = 2$: $b(v) = 6$;

- if $c(v) = 4$ and $c(mn(v)) = 3$: $b(v) = 2$.

(b) We construct a planar graph G_2 as follows. First we make three copies $(H_1, M_1), (H_2, M_2), (H_3, M_3)$ of the pair (H, M) from Figure 4.2(a), and glue them together at vertex v. Then we add one new vertex w and four new edges, i.e., the edge vw and the edges u_1u_2, u_2u_3, u_3u_1 (see Figure 4.2(b)). Let M' be a matching backbone of G_2 that contains all matchings M_i ($i = 1, 2, 3$) and the edge vw.

Suppose that there exists a 5-cyclic 2-backbone coloring of (G_2, M'). Without loss of generality we may assume that vertex v is colored with color 1. Then, by Lemma 4.4.3, vertices u_1, u_2 and u_3 must all be colored with either color 3 or color 4. On the other hand, the vertices u_1, u_2 and u_3 can not be colored with colors $1, 2$ and 5. On the other hand, vertices u_1, u_2 and u_3 must all get different colors, since they induce a K_3. This contradiction completes the proof of Theorem 4.4.4. □

4.5 λ-Backbone coloring numbers of split graphs

We recall that a *split graph* is a graph whose vertex set can be partitioned into a *clique* (i.e. a set of mutually adjacent vertices) and an *independent set* (i.e. a set of mutually nonadjacent vertices), with possibly edges in between. The size of a largest clique in G and the size of a largest independent set in G are denoted by $\omega(G)$ and $\alpha(G)$, respectively. Split graphs were introduced by Hammer & Földes [26]; see also the book [21] by Golumbic.

In this section we discuss the special case of λ-backbone colorings of split graphs with star backbones or matching backbones or tree backbones. The motivation for looking at split graphs is threefold. First of all, split graphs have nice structural properties. They form an interesting subclass of the class of perfect graphs. Hence, split graphs satisfy $\chi(G) = \omega(G)$. Secondly, every graph can be turned into a split graph by considering any (e.g. a maximum) independent set and turning the remaining vertices into a clique. Thirdly, the number of colors needed to color the resulting split graph is an upper bound for the number of colors one needs to color the original graph. It will become clear from the results below that split graphs indeed serve us very well in this specific context, since they can provide considerably lower upper bounds on the numbers of colors we need than earlier results.

4.5.1 Star backbones of split graphs

In this subsection we present sharp upper bounds for the λ-backbone coloring numbers of split graphs with star backbones. The following theorem is a strengthening of Theorem 4.2.1 for the special case of split graphs.

Theorem 4.5.1. *Let $\lambda \geq 2$ and let $G = (V, E)$ be a split graph with $\chi(G) = k \geq 2$. For every star backbone $S = (V, E_S)$ of G,*

$$\mathrm{BBC}_\lambda(G, S) \leq \begin{cases} k + \lambda & \text{if either } k = 3 \text{ and } \lambda \geq 2 \text{ or } k \geq 4 \text{ and } \lambda = 2 \\ k + \lambda - 1 & \text{in the other cases.} \end{cases}$$

The bounds are tight.

Proof. We divide the proof into two parts as follows.

Part 1 Proof of the upper bounds.
Let $G = (V, E)$ be a split graph with a star backbone $S = (V, E_S)$. Let C and I be a partition of V such that C with $|C| = k$ is a clique of maximum size, and such that I is an independent set. Then $\chi(G) = \omega(G) = k$. Let r_C and r_I be the number of roots in C and the number of roots in I, respectively. We define the root of any $S_1 \in S$ as the end vertex in C.

First, we consider the case that either $k = 3$ and $\lambda \geq 2$ or $k \geq 4$ and $\lambda = 2$. If $r_C = 1$ and $r_I = 0$, then color the root with color 1; color the other vertices in C with colors $1 + \lambda, \ldots, k + \lambda - 1$; color all vertices in I with color $k + \lambda$. If $r_C \neq 1$ or $r_I \geq 1$, then let p be the number of leaves in C of the roots in I. Color the roots in C with colors $1, \ldots, r_C$. Color all the leaves in C of the roots in I with colors $r_C + 1, \ldots, r_C + p$. Color the other vertices in C with colors $\lambda + r_C + p, \ldots, k + \lambda - 1$. Color all vertices in I with color $k + \lambda$. This results in a λ-backbone coloring with colors from $\{1, \ldots, \chi(G) + \lambda\}$.

Next, for $k = 2$ and $\lambda \geq 2$ color the two vertices in C with colors 1 and $\lambda + 1$. Color every vertex $u \in I$ with color $\lambda + 1$ if u is a leaf of a star with 1 as its root color, and color every vertex $u \in I$ with color 1 if u is a leaf of a star with $\lambda + 1$ as its root color. This results in a λ-backbone coloring with colors from $\{1, \ldots, \chi(G) + \lambda - 1\}$.

Finally, for $k \geq 4$ and $\lambda \geq 3$, we distinguish nine cases which are based on the number and the location of the roots, and we indicate how a suitable λ-backbone coloring is obtained.

Case 1 $r_C = 0$.
Let w be the number of leaves of a root that has the largest number of leaves.

Color the leaves of one root that has w leaves with colors $2, \ldots, w+1$, and color their root with color $k+\lambda-1$. Color the other roots with color 1. Color the other vertices in C with colors $w+\lambda-1, \ldots, k+\lambda-2$.

Case 2 $r_C = 1$ and either $r_I = 0$ or $r_I = 1$ and all leaves of the root in C are in I.
Color each root with color $k+\lambda-1$. Note that in case $r_C = 1$ and $r_I = 1$ the roots are nonadjacent since $|C|$ is maximum and all other vertices of C are leaves of the root in I. Color the $k-1$ leaves in C with colors $1, \ldots, k-1$. Each leaf $u \in I$ is colored with $min\{\text{color of } v | v \in C, uv \notin E\}$.

Case 3 $r_C = 1$, $r_I = 1$ and the root in C has at least one leaf in C.
Color the root in C with color 1. Let t be the number of leaves in C of the root in C. Color the t leaves in C of the root in C with colors $\lambda+k-t-1, \ldots, \lambda+k-2$. Color the $k-t-1$ leaves in C of the root in I with colors $2, \ldots, k-t$. Color all vertices in I with color $\lambda+k-1$.

Case 4 $r_C = 1$ and $r_I \geq 2$.
Color the root in C with color 2. Let w be the number of leaves of a root in I which has the largest number of leaves. Color all the leaves of one root in I which has w leaves with colors $k+\lambda-1-w, \ldots, k+\lambda-2$, and color the root with color 1. Let x be the number of leaves in C who have their roots in C. Color the leaves in C who have their roots in C with colors $k+\lambda-1-w-x, \ldots, k+\lambda-2-w$. Color the other vertices in C with colors $3, \ldots, k+1-w-x$. Color all other vertices in I with color $k+\lambda-1$.

Case 5 $r_C = 2$ and $r_I = 0$.
Color a root which has the largest number of leaves in C with color 1, and the other root with color 2. Use color $\lambda+1$ for one leaf in C of the root with color 1. Color the $k-3$ other leaves in C with colors $\lambda+2, \ldots, \lambda+k-2$. Color all leaves in I with color $\lambda+k-1$.

Case 6 $r_C = 2$ and $r_I \geq 1$.
Color the two roots in C with colors 2 and 3. Let w be the number of leaves of a root in I which has the largest number of leaves. Color all the leaves of one root in I which has w leaves with colors $\lambda+1, \ldots, \lambda+w$, and color the root with color 1. Let x be the number of leaves in C who have their roots in C. Color all the leaves in C who have their roots in C with colors $\lambda+w+1, \ldots, \lambda+w+x$. Color the other vertices in C with colors $4, \ldots, k+1-w-x$. Color all other vertices in I with color $k+\lambda-1$.

Case 7 $3 \leq r_C \leq k-2$ for $k \geq 5$.
Color all roots in I with color 1. Color the r_C roots in C with colors $2, \ldots, r_C+1$ consecutively based on the number of their leaves in C from the largest one to the smallest one. Color the $k-r_C$ leaves in C with colors $r_C+\lambda-1, \ldots, k+\lambda-2$ consecutively based on the color of their root from the smallest one to the largest one. Color all leaves in I with color $\lambda+k-1$.

Case 8 $r_C = k-1$.
Then $r_I = 0$. Color one root and its leaf in C with colors 2 and $k+\lambda-2$, respectively. Color the $k-2$ other roots in C with colors $3, \ldots, k-1$ and $k+\lambda-3$. Color the leaves in I which have the root with color $k+\lambda-3$ with color 1. Color the other leaves in I with color $k+\lambda-1$.

Case 9 $r_C = k$.
Then $r_I = 0$. Color the k roots with colors $2, \ldots, k-1, k+\lambda-3, k+\lambda-2$. Color all leaves in I whose root has color $k+\lambda-3$ or $k+\lambda-2$ with color 1. Color the other leaves in I with color $k+\lambda-1$.

Part 2 Proof of the tightness of the bounds.
For the case $k = 3$ and $\lambda \geq 2$ and the case $k \geq 4$ and $\lambda = 2$ we consider a split graph $G = (V, E)$ with a clique of k vertices $v_1, \ldots, v_k$ and with an independent set of $k(k-1)$ vertices $u_{i,j}$ with $1 \leq i \leq k$, $1 \leq j \leq k$ and $i \neq j$. Every vertex $u_{i,j}$ is adjacent to all vertices v_s with $s \neq j$. The star backbone S contains the $k(k-1)$ edges $u_{i,j}v_i$ with $1 \leq i \leq k$, $1 \leq j \leq k$ and $i \neq j$. So, all vertices in the clique are all the roots of S. Clearly, $\chi(G) = k$. Suppose to the contrary that $\text{BBC}_\lambda(G, S) \leq k+\lambda-1$, and consider such a backbone coloring. The vertices $v_1, \ldots, v_k$ in the clique must be colored with k pairwise distinct colors such that every color has at least one other color at distance λ.

For the case $k = 3$ and $\lambda \geq 2$ we only have four colors that are possible to be used for the three roots in the clique, i.e. $1, 2, \lambda+1, \lambda+2$. Hence, there are four choices to color the clique. But by symmetry, we only have to check two of them. The first case is that we use colors $1, 2, \lambda+1$ for the vertices in the clique. All leaves of the root with color $\lambda+1$ must be colored with color 1. We find a contradiction, since there is one leaf of the root with color $\lambda+1$ that is adjacent in G with the root with color 1. The second case is that we use colors $1, 2, \lambda+2$ for the vertices in the clique. All leaves of the root with color 2 must be colored with color $\lambda+2$. We find a contradiction, since there is a leaf of the root with color 2 that is adjacent in G with the root with color $\lambda+2$.

Next, we consider the case $k \geq 4$ and $\lambda = 2$. Suppose that we use colors from

$\{1, \ldots, k+1\} \setminus \{i\}$ for some $i = k$ or $k+1$ for the k vertices in the clique. Let v_l and v_m be the roots with color $i-1$ and color $i-2$, respectively. Each leaf of the root v_l must be colored with one of the $k-2$ colors in $\{1, \ldots, k+1\} \setminus \{i, i-1, i-2\}$. Since $u_{l,m}v_s \in E$ for $s = 1, \ldots, k$ and $s \neq m$, we can not color the vertex $u_{l,m}$. We find a contradiction. A similar argument can be used for the other possibilities. Suppose that we use colors from $\{1, \ldots, k+1\} \setminus \{i\}$ for some $i = 1, \ldots, k-1$ for the k vertices in the clique. Let v_y and v_z be the roots with color $i+1$ and color $i+2$, respectively. Each leaf of the root v_y must be colored with one of the $k-2$ colors in $\{1, \ldots, k+1\} \setminus \{i, i+1, i+2\}$. Since $u_{y,z}v_s \in E$ for $s = 1, \ldots, k$ and $s \neq z$, we can not color the vertex $u_{y,z}$. We find a contradiction.

For the remaining case we consider a complete graph G with k vertices $v_1, \ldots, v_k$. The star backbone S contains the $k-1$ edges v_kv_s with $1 \leq s \leq k-1$. Clearly, $\chi(G) = k$. Since the vertices $v_1, \ldots, v_k$ are in the clique and since the vertices $v_1, \ldots, v_{k-1}$ are the leaves of the root v_k, we need at least $k-1+\lambda$ colors in a λ-backbone coloring of (G, S). □

4.5.2 Matching backbones of split graphs

In this subsection we present sharp upper bounds for the λ-backbone coloring numbers of split graphs with matching backbones in Theorem 4.5.4. It is a strengthening of Theorem 4.3.1 for the special case of split graphs. Before we present the complete results about it, we introduce the notion of matching neighbor, nonneighbor and splitting set in a split graph with a matching backbone, and we prove two technical lemmas (Lemma 4.5.2 and Lemma 4.5.3).

Given a split graph $G = (V, E)$ with a matching backbone $M = (V, E_M)$. A vertex $u \in V$ is called a *matching neighbor* of vertex $v \in V$ if $(u, v) \in E_M$, denoted by $u = mn(v)$. Let C be the largest clique of G and let I be the largest independent set of G. A *set of nonneighbors of an element* $u \in C$ is defined as the set of vertices $v \in I$ for which $(u, v) \notin E$. Similarly, a *set of nonneighbors of an element* $v \in I$ is defined as the set of vertices $u \in C$ for which $(v, u) \notin E$. The set of nonneighbors of a vertex u is denoted by $nn(u)$. Note that every vertex in I has at least one nonneighbor. However, for a vertex $u \in C$, the set $nn(u)$ may be empty. For some $p \leq \alpha(G)$ a *splitting set of cardinality* p, named an s-set for short, is defined as a set $\{v_1, \ldots, v_p\} \subseteq I$

such that

$$\left\{ \bigcup_{i=1,\ldots,p} nn(v_i) \right\} \bigcap \left\{ \bigcup_{i=1,\ldots,p} mn(v_i) \right\} = \emptyset.$$

Note that if (G, M) has an s-set of cardinality p, then it also has an s-set of cardinality q for all $q \leq p$.

Lemma 4.5.2. *Given a split graph $G = (V, E)$ with a matching backbone $M = (V, E_M)$. Let k' be the cardinality of a clique C' in G and let i' be the cardinality of an independent set I' in G. If $i' = k'$, every vertex in I' has at most one nonneighbor in C' and has exactly one matching neighbor in C', and $\lceil \frac{k'}{3} \rceil \geq x$, then (G, M) has an s-set of cardinality x that is a subset of I'.*

Proof. We split C' and I' up in C'_1, C'_2, I'_1 and I'_2 with cardinality c'_1, c'_2, i'_1 and i'_2, respectively, in the following way.

- C'_1 consists of all the vertices in C' that either have no nonneighbors in I' or have at least two nonneighbors in I' or have exactly one nonneighbor in I', whose matching neighbor in C' has no nonneighbors in I'.
- C'_2 consists of all other vertices in C'. Obviously, they all have exactly one nonneighbor in I'.
- I'_1 consists of the matching neighbors of the vertices in C'_1.
- I'_2 consists of the matching neighbors of the vertices in C'_2.

Clearly, $i'_1 = c'_1$ and $i'_2 = c'_2$. Now assume that there are ℓ vertices in C'_1 that have no nonneighbors in I' and put them in set L. Also assume that there are q vertices in C'_1 that have at least two nonneighbors in I' and put them in set Q. Finally, assume that there are n vertices in C'_1 that have exactly one nonneighbor in I', whose matching neighbor has no nonneighbors in I' and put them in set N. Then $\ell \geq q$, $\ell \geq n$ and $c'_1 = \ell + q + n$, so $c'_1 \leq 3\ell$.

Let L', Q' and N' be the sets of matching neighbors of the vertices in L, Q and N, respectively. We pick from I'_1 the ℓ vertices in L' and put them in the s-set. Notice that these vertices do not violate the definition of an s-set, because the set of their nonneighbors and the set of their matching neighbors are two disjoint sets. The matching neighbors of the nonneighbors of the ℓ vertices in the s-set are either in Q' or in N', so we exclude the vertices in these two sets for use in the s-set. On the other hand, the matching neighbors of the ℓ vertices in the s-set do not have nonneighbors, so we do not have to worry about them. From the observations above it is clear that we can pick $l \geq \lceil \frac{c'_1}{3} \rceil = \lceil \frac{i'_1}{3} \rceil$ vertices from I'_1 that can be used in the s-set. Moreover, any vertices from I'_2 that are put in the s-set do not conflict with the vertices from L' that are in the s-set already. So the only thing we have to do now is to pick at least $\lceil \frac{i'_2}{3} \rceil$ vertices from I'_2 that can be used in the s-set. We have to verify again that these vertices do not violate the definition of the s-set. Pick an arbitrary vertex from I'_2 and put it in the s-set. Now delete from I'_2 the matching neighbor of its nonneighbor and the unique nonneighbor of its matching neighbor if they happen to be in I'_2. Continuing this way, we lose at most two vertices of I'_2 for every vertex of I'_2 that we put in the s-set. So we can pick at least $\lceil \frac{i'_2}{3} \rceil$ vertices from I'_2 that we can put in the s-set. Therefore, the cardinality of the s-set is at least $\lceil \frac{i'_1}{3} \rceil + \lceil \frac{i'_2}{3} \rceil \geq \lceil \frac{i'}{3} \rceil = \lceil \frac{k'}{3} \rceil \geq x$, which proves the lemma. □

Lemma 4.5.3. *Given a split graph $G = (V, E)$ with a matching backbone $M = (V, E_M)$. Let $k = \omega(G)$ be the cardinality of the largest clique C in G and let $i = \alpha(G)$ be the cardinality of the largest independent set I in G. If every vertex in I has exactly one nonneighbor in C and $\lceil \frac{k}{3} \rceil \geq x$, then (G, M) has an s-set S with $|S| = x - \frac{k-i}{2}$ such that there are no matching edges between the nonneighbors of vertices of S.*

Proof. To prove this lemma, we first define three disjoint subsets of C.

- C_1 consists of the i vertices of C that have a matching neighbor in I.
- C_2 contains for each matching edge in C for which both vertices have at least one nonneighbor in I, the vertex with the fewest nonneighbors in I. If both vertices have the same number of nonneighbors in I, then any one vertex will be in C_2.
- C_3 contains for each matching edge in C for which both vertices have at least one nonneighbor in I, the vertex that is not in C_2.

Let m be the sum of the number of nonneighbors of the vertices in C_2 and let n be the sum of the number of nonneighbors of vertices in C_3. Then clearly, $n \geq m$ and there are at least $m+n$ vertices in C_1 that have no nonneighbors in I. We give a partition of I into four sets, $I_1, \ldots, I_4$ and we show that one can pick n vertices from I_2 and at least $x - \frac{k-i}{2} - n$ vertices from I_4 that together will form the s-set.

- I_1 consists of all the nonneighbors of the vertices in C_2.
- I_2 consists of the matching neighbors of n vertices in C_1 that have no nonneighbors in I and whose matching neighbors are not already in I_1.
- I_3 consists of the matching neighbors of the nonneighbors of the elements of I_2, that are in I, but not in I_1.
- I_4 consists of the other vertices of I.

Let i_1, i_2, i_3 and i_4 be the cardinality of I_1, I_2, I_3 and I_4, respectively. It is easily verified that $i_1 = m,\ \ i_2 = n,\ \ i_3 \leq n$ and $i_4 \geq i - (2n + m)$. Now we put all the vertices of I_2 in the s-set and leave all the vertices of I_1 and I_3 out of the s-set. Since the vertices of I_1 are not in the s-set, there are no matching edges between the nonneighbors of vertices in the s-set. Since the matching neighbors of the vertices that are now in the s-set have no nonneighbors, and the matching neighbors of the nonneighbors of the vertices in the s-set are not in I_4, vertices from I_4 that will be added to the s-set do not conflict with vertices from I_2 that are already there. Now consider the set I_4 and the set C_4 of its matching neighbors in C as an independent set and a clique of the graph G with the matching backbone M. Clearly, every vertex in I_4 has at most one nonneighbor in C_4 and exactly one matching neighbor in C_4. Moreover, if c_4 is the cardinality of the clique C_4, then $i_4 = c_4$ and $\lceil \frac{c_4}{3} \rceil = \lceil \frac{i_4}{3} \rceil \geq \lceil \frac{k-(k-i)-(2n+m)}{3} \rceil \geq \lceil \frac{k}{3} \rceil - \lceil \frac{k-i}{3} \rceil - \lceil \frac{2n+m}{3} \rceil \geq x - \lceil \frac{k-i}{2} \rceil - n = x - \frac{k-i}{2} - n$. Thus, by Lemma 4.5.2, (G, M) has an s-set of cardinality $x - \frac{k-i}{2} - n$ that is a subset of I_4. Therefore, we can add these $x - \frac{k-i}{2} - n$ vertices from I_4 to the s-set of (G, M). Together with the n vertices from I_2 that were already in there, we obtain that (G, M) has an s-set of cardinality $x - \frac{k-i}{2}$ such that there are no matching edges between the nonneighbors of vertices of the s-set. □

Theorem 4.5.4. *Let $\lambda \geq 2$ and let $G = (V, E)$ be a split graph with $\chi(G) = k \geq 2$. For every matching backbone $M = (V, E_M)$ of G,*

$$\text{BBC}_\lambda(G, M) \leq \begin{cases} \lambda + 1 & \text{if } k = 2 \\ k + 1 & \text{if } k \geq 3 \text{ and } \lambda \leq \min\{\frac{k}{2}, \frac{k+5}{3}\} \\ k + 2 & \text{if } k = 9 \text{ or } k \geq 11 \text{ and } \frac{k+6}{3} \leq \lambda \leq \lceil \frac{k}{2} \rceil \\ \lceil \frac{k}{2} \rceil + \lambda & \text{if } k = 3, 5, 7 \text{ and } \lambda \geq \lceil \frac{k}{2} \rceil \\ \lceil \frac{k}{2} \rceil + \lambda + 1 & \text{if } k = 4, 6 \text{ or } k \geq 8 \text{ and } \lambda \geq \lceil \frac{k}{2} \rceil + 1. \end{cases}$$

The bounds are tight.

Proof. First of all, note that for technical reasons we split up the proof in more and different subcases than there appear in the formulation of the theorem. We divide the proof into two parts as follows.

Part 1 Proof of the upper bounds.
If $k = 2$, then G is bipartite, and we use colors 1 and $\lambda + 1$. For $k \geq 3$, let $G = (V, E)$ be a split graph with $\chi(G) = k$ and with a matching backbone $M = (V, E_M)$. Let C and I be a partition of V such that C with $|C| = k$ is a clique of maximum size, and such that I with $|I| = i$ is an independent set. Without loss of generality, we assume that every vertex in I has exactly one nonneighbor in C.

First, we consider the case $k = 4, 6, 8, 10$ and $\lambda \leq \frac{k}{2}$, the case $k = 2m$, $m \geq 6$ and $\lambda \leq \frac{k+5}{3}$, and the case $k = 2m + 1$, $m \geq 1$ and $\lambda \leq \frac{k+5}{3}$. For these cases we obtain $k \geq 2\lambda - 1$ and $\lceil \frac{k}{3} \rceil \geq \lambda - 1$. By Lemma 4.5.3, we find that (G, M) has an s-set of cardinality $y = \lambda - 1 - \frac{k-i}{2}$ such that there are no matching edges between the nonneighbors of vertices of the s-set. We make a partition of C into six disjoint sets, $C_1, \ldots, C_6$, with cardinality $c_1, \ldots, c_6$, respectively.

- C_1 consists of those vertices in C that have a matching neighbor in C and a nonneighbor in the s-set. Notice that, by definition of the s-set, there are no matching edges between vertices in C_1.

- C_2 consists of those vertices in C that have a matching neighbor in I and a nonneighbor in the s-set.

- C_3 contains one vertex of each matching edge in C that has no vertex in C_1.

- C_4 consists of those vertices in C whose matching neighbor is in I and that are neither matching neighbor nor nonneighbor of any vertex in the s-set.

- C_5 consists of those vertices in C that have a matching neighbor in the s-set.

- C_6 consists of those vertices in C that have a matching neighbor in C and that are not already in C_1 or C_3.

It is easily verified that

$$c_1 + c_2 \leq y, \qquad c_3 = \tfrac{k-i}{2} - c_1, \qquad c_4 = i - y - c_2,$$
$$c_5 = y, \qquad c_6 = \tfrac{k-i}{2}, \qquad \textstyle\sum_{i=1}^{6} c_i = k.$$

An algorithm that constructs a feasible λ-backbone coloring of (G, M) with at most $k+1$ colors is given on the next page. In this algorithm and later on, I'' denotes the set consisting of the vertices of I that are not in the s-set.

Coloring Algorithm 1

1 Color the vertices in C_1 with colors from the set $\{1, \ldots, c_1\}$.

2 Color the vertices in C_2 with colors from the set $\{c_1 + 1, \ldots, c_1 + c_2\}$.

3 Color the vertices in the s-set by assigning to them the same colors as their nonneighbors in C_1 or C_2. Note that different vertices in the s-set can have the same nonneighbor in C_1 or C_2, so a color may occur more than once in the s-set.

4 Color the vertices in C_3 with colors from the set $\{c_1 + c_2 + 1, \ldots, c_1 + c_2 + c_3\}$.

5 Color the vertices in C_4 with colors from the set $\{c_1+c_2+c_3+1, \ldots, c_1+c_2+c_3+c_4\}$.

6 Color the vertices in C_5 with colors from the set $\{c_1 + c_2 + c_3 + c_4 + 1, \ldots, c_1+c_2+c_3+c_4+c_5\}$; start with assigning the lowest color from this set to the matching neighbor of the vertex in the s-set with the lowest color and continue this way.

7 Color the vertices in C_6 with colors from the set $\{c_1 + c_2 + c_3 + c_4 + c_5 + 1, \ldots, c_1 + c_2 + c_3 + c_4 + c_5 + c_6\}$; start with assigning the lowest color from this set to the matching neighbor with the lowest color in $C_1 \cup C_3$ and continue this way.

8 Finally, color the vertices of I'' with color $k + 1$.

It is clear that all the vertices in C get different colors, and that vertices in I either get a color that does not occur in C or get the same color as their nonneighbor in C. There are three types of matching edges for which we have to verify that the distance between the colors of their vertices is at least λ:

1. Matching edges in C. They have one vertex in $C_1 \cup C_3$ and the other vertex in C_6. It is easy to see that the smallest distance between two colors here occurs for the matching edges that have one vertex in C_3 and the other vertex in C_6. This distance is $c_4 + c_5 + c_6 = i - c_2 + \frac{k-i}{2} \geq i - y + \frac{k-i}{2} = i - \lambda + 1 + \frac{k-i}{2} + \frac{k-i}{2} = k - \lambda + 1 \geq 2\lambda - 1 - \lambda + 1 = \lambda$.

2. Matching edges between the s-set and C. These are exactly y matching edges. They have one vertex in the s-set and the other vertex in C_5, so one vertex gets a color from the set $\{1, \ldots, c_1 + c_2\}$ and the other vertex gets a color from the set $\{c_1 + c_2 + c_3 + c_4 + 1, \ldots, c_1 + c_2 + c_3 + c_4 + c_5\}$. This last set contains exactly y colors, but the first set may contain fewer than y colors, so some of the colors of the first set may be used more than once in the s-set. However, it is not hard to see that the smallest distance between two colors here occurs for the matching edge with colors 1 and $c_1+c_2+c_3+c_4+1$. This distance is equal to $c_1+c_2+c_3+c_4 = k-c_5-c_6 = k - y - \frac{k-i}{2} = k - \lambda + 1 + \frac{k-i}{2} - \frac{k-i}{2} = k - \lambda + 1 \geq 2\lambda - 1 - \lambda + 1 = \lambda$.

3. Matching edges between I'' and C. They have one vertex in I'' and the other vertex in $C_2 \cup C_4$. It is clear that the smallest distance between two colors for a matching edge of this type is equal to $k+1-c_1-c_2-c_3-c_4 = c_5 + c_6 + 1 = y + \frac{k-i}{2} + 1 = \lambda - 1 - \frac{k-i}{2} + \frac{k-i}{2} + 1 = \lambda$.

It shows that the coloring provided by Coloring Algorithm 1 is a λ-backbone coloring of (G, M) with colors from $\{1, \ldots, k+1\}$.

Next, we consider the case $k = 2m$, $m \geq 6$ and $\frac{k+6}{3} \leq \lambda \leq \frac{k}{2}$. We obtain $k \geq 2\lambda$. We color the k vertices in C with colors from the sets $\{2, \ldots, \frac{k}{2} + 1\}$ and $\{\frac{k}{2} + 2, \ldots, k+1\}$. If there are matching edges in C, then we assign the first colors from both sets to the two vertices of the first matching edge, the second colors from both sets to the two vertices of the second matching edge and so on. We can color up the two vertices of $\frac{k}{2}$ matching edges in C this way and this is the maximum number of matching edges in C. Vertices in I get color $k+2$ if their matching neighbor in C is colored by a color from the first set, and vertices in I get color 1 if their matching neighbor in C is colored by a color from the second set. This results in a λ-backbone coloring of (G, M) with at most $k+2$ colors.

We consider the case $k = 2m+1$, $m \geq 4$ and $\frac{k+6}{3} \leq \lambda \leq \frac{k+1}{2}$. We obtain $k \geq 2\lambda - 1$. For this case we find that i is odd, otherwise there is no perfect matching of G. If $i = 1$, then there are $\frac{k-1}{2}$ matching edges in C. We can color their vertices with colors from the two sets $\{1, \ldots, \frac{k-1}{2}\}$ and $\{\frac{k-1}{2} + 3, \ldots, k+1\}$, such that the first colors from both sets are assigned to the two vertices of one matching edge, the second colors from both sets are assigned to the two vertices of another matching edge and so on. The distance between two colors of the two vertices in every matching edge in C is $\frac{k-1}{2}+2 \geq \frac{2\lambda-2}{2}+2 = \lambda+1$. For the other vertex in C we use color $\frac{k-1}{2}+1$ and its matching neighbor in I gets color $k+2$. Note that $k+2-\frac{k-1}{2}-1 = \frac{k+3}{2} \geq \frac{2\lambda+2}{2} = \lambda+1$. If i is odd and $3 \leq i \leq k$, then there are $\frac{k-i}{2}$ matching edges in C. We can color their vertices with colors from the two sets $\{2,\ldots,\frac{k-i}{2}+1\}$ and $\{\frac{k+i}{2}+2,\ldots,k+1\}$, such that the first colors from both sets are assigned to the two vertices of one matching edge, the second colors from both sets are assigned to the two vertices of another matching edge and so on. The distance between two colors of the two vertices in every matching edge in C is $\frac{k+i}{2} \geq \frac{2\lambda-1+i}{2} \geq \frac{2\lambda+2}{2} = \lambda+1$. The other i vertices in C are colored with colors from the sets $\{\frac{k-i}{2}+2,\ldots,\frac{k}{2}+1\}$ and $\{\lambda+1,\ldots,\frac{i}{2}+\lambda\}$, which are exactly i colors. Vertices in I get color $k+2$ if their matching neighbor in C is colored by a color from the first set, or get color 1 if their matching neighbor in C is colored by a color from the second set. Notice that $k+2-\frac{k+3}{2} = \frac{2k+4-k-3}{2} = \frac{k+1}{2} \geq \frac{2\lambda}{2} = \lambda$ and $\frac{k+3}{2}+1-1 = \frac{k+3}{2} \geq \frac{2\lambda+2}{2} = \lambda+1$, so all these matching edges have the required distance of at least λ. This results in a λ-backbone coloring of (G,M) with at most $k+2$ colors.

Next, we consider the case $k = 3,5,7$ and $\lambda \geq \frac{k+6}{3}$. We obtain $\lceil\frac{k}{3}\rceil \geq \frac{k-1}{2}$. By Lemma 4.5.3, we find that (G,M) has an s-set of cardinality $z = \frac{k-1}{2} - \frac{k-i}{2} = \frac{i-1}{2}$ such that there are no matching edges between the nonneighbors of vertices of the s-set. We have to construct a λ-backbone coloring of (G,M) with at most $\frac{k+1}{2}+\lambda$ colors. Obviously, colors from the set $\{\frac{k+1}{2}+1,\ldots,\lambda\}$ can not be used at all. So we have to find a coloring with colors from the sets $\{1,\ldots,\frac{k+1}{2}\}$ and $\{\lambda+1,\ldots,\frac{k+1}{2}+\lambda\}$. We split C up in 6 different sets in the way we did this in the proof of the case $k = 4,6,8,10$ and $\lambda \leq \frac{k}{2}$.

For the cardinality of these sets, we have the following relations:

$$c_1 + c_2 \leq \frac{i-1}{2}, \qquad c_3 = \frac{k-i}{2} - c_1, \qquad c_4 = i - \frac{i-1}{2} - c_2,$$
$$c_5 = \frac{i-1}{2}, \qquad c_6 = \frac{k-i}{2}, \qquad \sum_{i=1}^{6} c_i = k.$$

The following variation on Coloring Algorithm 1 constructs a feasible λ-backbone coloring of (G,M).

Coloring Algorithm 2

1 - 5 are the same as in Coloring Algorithm 1.

6 Color the vertices in C_5 with colors from the set $\{\lambda+1,\ldots,\lambda+c_5\}$; start with assigning the lowest color from this set to the matching neighbor of the vertex in the s-set with the lowest color and continue this way.

7 Color the vertices in C_6 with colors from the set $\{\lambda+c_5+1,\ldots,\lambda+c_5+c_6\}$; start with assigning the lowest color from this set to the matching neighbor with the lowest color in $C_1 \cup C_3$ and continue this way.

8 Finally, color the vertices in I'' with color $\frac{k+1}{2}+\lambda$.

Again, it is clear that vertices in C all get different colors and that vertices in I either get a color that does not occur in C or get the same color as their nonneighbor in C. Also again, there are three types of matching edges for which we have to verify that the distance between their vertices is at least λ:

1. Matching edges in C. They have one vertex in $C_1 \cup C_3$ and one vertex in C_6. It is easy to see that the smallest distance between two colors here occurs for the matching edges that have one vertex in C_3 and the other vertex in C_6. This distance is $\lambda+c_5+c_6-c_1-c_2-c_3=\lambda+\frac{i-1}{2}+\frac{k-i}{2}-\frac{k-i}{2}-c_2 \geq \lambda+\frac{i-1}{2}-\frac{i-1}{2}=\lambda$.

2. Matching edges between the s-set and C. These are exactly $z=\frac{i-1}{2}$ matching edges. They have one vertex in the s-set and the other vertex in C_5, so one vertex gets a color from the set $\{1,\ldots,c_1+c_2\}$ and the other vertex gets a color from the set $\{\lambda+1,\ldots,\lambda+c_5\}$. This last set contains exactly z colors, but the first set may contain fewer than z colors, so some of the colors of the first set may be used more than once in the s-set. However, it can be verified that the smallest distance between two colors here occurs for the matching edge with colors 1 and $\lambda+1$. This distance is equal to λ.

3. Matching edges between I'' and C. They have one vertex in I'' and the other vertex in $C_2 \cup C_4$. It is clear that the smallest distance between two colors for a matching edge of this type is equal to $\frac{k+1}{2}+\lambda-c_1-c_2-c_3-c_4=\frac{k+1}{2}+\lambda-\frac{k-i}{2}-i+\frac{i-1}{2}=\lambda+\frac{k+1-k+i-2i+i-1}{2}=\lambda$.

It shows that the coloring provided by Coloring Algorithm 2 is a λ-backbone coloring of (G,M) with colors from $\{1,\ldots,\frac{k+1}{2}+\lambda\}$.

Next, we consider the case $k=2m$, $m \geq 2$ and $\lambda \geq \frac{k}{2}+1$. For this case we find that i is even, otherwise there is no perfect matching of G. If $i=0$, then there are $\frac{k}{2}$ matching edges in C. We can use color pairs $\{1, \lambda + 1\}, \{2, \lambda + 2\}, \ldots, \{\frac{k}{2}, \frac{k}{2} + \lambda\}$ for their vertices, because $\lambda + 1 > \frac{k}{2}$. If i is even and $i \geq 2$, then there are $\frac{k-i}{2}$ matching edges in C. We can color their vertices with colors from the two sets $\{2, \ldots, \frac{k-i}{2}+1\}$ and $\{\frac{i}{2}+\lambda+1, \ldots, \frac{k}{2}+\lambda\}$, such that the first colors from both sets are assigned to the two vertices of one matching edge, the second colors from both sets are assigned to the two vertices of another matching edge and so on. The distance between two colors of the two vertices in every matching edge in C is $\frac{i}{2} + \lambda - 1 \geq \lambda$. The other i vertices in C are colored with colors from the sets $\{\frac{k-i}{2} + 2, \ldots, \frac{k}{2} + 1\}$ and $\{\lambda + 1, \ldots, \frac{i}{2} + \lambda\}$, which are exactly i colors. The colors in this first set have distance at least λ to color $\frac{k}{2} + \lambda + 1$, so we color the matching neighbors in I of the vertices in C that are colored with colors from this first set with color $\frac{k}{2} + \lambda + 1$. The colors in the second set have distance at least λ to color 1, so we color the matching neighbors in I of the vertices in C that are colored with colors from this second set with color 1. This results in a λ-backbone coloring of (G, M) with at most $\frac{k}{2} + \lambda + 1$ colors.

Finally, we consider the case $k = 2m+1$, $m \geq 4$ and $\lambda \geq \frac{k+1}{2}+1$. For this case we find that i is odd, otherwise there is no perfect matching of G. There are $\frac{k-i}{2}$ matching edges in C. We can color their vertices with colors from the two sets $\{2, \ldots, \frac{k-i}{2}+1\}$ and $\{\frac{i+3}{2}+\lambda, \ldots, \frac{k+1}{2}+\lambda\}$, such that the first colors from both sets are assigned to the two vertices of one matching edge, the second colors from both sets are assigned to the two vertices of another matching edge and so on. Notice that $\frac{i+3}{2} + \lambda - \frac{k-i}{2} - 1 = \frac{i+3+2\lambda-k+i-2}{2} = \frac{2i+1-k+2\lambda}{2} \geq \frac{2i+1-k+k+2}{2} > 0$, so that these sets have no overlap. The distance between two colors of the two vertices in every matching edge in C is $\frac{i-1}{2} + \lambda \geq \lambda$. The other i vertices in C are colored with colors from the sets $\{\frac{k-i}{2} + 2, \ldots, \frac{k+1}{2}\}$ and $\{\lambda + 1, \ldots, \frac{i+1}{2} + \lambda\}$, which are exactly i colors that have not been used yet. Vertices in I get color $\frac{k+1}{2} + \lambda + 1$ if their matching neighbor in C is colored by a color from the first set, or get color 1 if their matching neighbor in C is colored by a color from the second set. This results in a λ-backbone coloring of (G, M) with at most $\frac{k+1}{2} + \lambda + 1$ colors.

Part 2 Proof of the tightness of the bounds.
The case $k = 2$ is trivial. For the case $k = 4, 6, 8, 10$ and $\lambda \leq \frac{k}{2}$, the case $k = 2m$, $m \geq 6$ and $\lambda \leq \frac{k+5}{3}$, the case $k = 2m + 1$, $m \geq 1$ and $\lambda \leq \frac{k+5}{3}$, the case $k = 3, 5, 7$ and $\lambda \geq \frac{k+6}{3}$, and the case $k = 2m$, $m \geq 2$ and $\lambda \geq \frac{k}{2} + 1$ we consider a split graph G with a clique of k vertices $v_1, \ldots, v_k$ and with an independent set of k vertices $u_1, \ldots, u_k$. Every vertex u_i with $i = 1, \ldots, k - 1$

is adjacent to all vertices v_j with $j = 1, \ldots, k-1$. The vertex u_k is adjacent to all vertices v_j with $j = 2, \ldots, k$. The matching backbone M contains the k edges $u_i v_i$ with $i = 1, \ldots, k$.

Suppose to the contrary that $\text{BBC}_\lambda(G, M) \le k$ for either the case $k = 4, 6, 8, 10$ and $\lambda \le \frac{k}{2}$, or the case $k = 2m$, $m \ge 6$ and $\lambda \le \frac{k+5}{3}$, or the case $k = 2m+1$, $m \ge 1$ and $\lambda \le \frac{k+5}{3}$. Then all k colors are used in the clique and the vertices u_i, with $i = 1, \ldots, k-1$, must get the same color as the color of v_k. We find a contradiction, since one color can be used at most $k - \lambda \le k-2$ times in the independent set.

Suppose to the contrary that $\text{BBC}_\lambda(G, M) \le \frac{k-1}{2} + \lambda$ for the case $k = 3, 5, 7$ and $\lambda \ge \frac{k+6}{3}$. Then colors from the set $\{\frac{k-1}{2} + 1, \ldots, \lambda\}$ can not be used at all, since these colors have no other colors at a distance of at least λ within the set $\{1, \ldots, \frac{k-1}{2} + \lambda\}$. Therefore, only the $k-1$ other colors can be used. We find a contradiction, since there is no way to color a clique of size k with only $k-1$ colors.

Suppose to the contrary that $\text{BBC}_\lambda(G, M) \le \frac{k}{2} + \lambda$ for the case $k = 2m$, $m \ge 2$ and $\lambda \ge \frac{k}{2} + 1$. Then colors from the set $\{\frac{k}{2} + 1, \ldots, \lambda\}$ can not be used at all, since these colors have no other colors at a distance of at least λ within the set $\{1, \ldots, \frac{k}{2} + \lambda\}$. Therefore, only the other k colors can be used. So all these k colors must occur in the clique and the vertices u_i, with $i = 1, \ldots, k-1$, must get the same color as the color of v_k. We find a contradiction, since one color can be used at most $\frac{k}{2} \le k-2$ times in the independent set.

Before completing the proof for the remaining cases we introduce the following definition. Let G be a split graph on $2k$ vertices with $k = \omega(G) = \alpha(G)$. The matching backbone M contains all edges between the largest clique C and the largest independent set I. Let every vertex in I have exactly one nonneighbor in C and let the matching edges together with the nonneighbor relations (see these nonneighbor relations as some imaginary edges) form one cycle of length $2k$. By $C_{k,k}$ we mean the representation of this split graph only by its vertices, its matching edges and the nonneighbor relations between C and I.

For the remaining cases we consider split graphs G with matching backbones M that are defined by the following characteristics.

1. $\omega(G) = \alpha(G)$,

2. $|nn(v)| = 1, \ \forall v \in I$,

3. The representation by their vertices, matching edges and nonneighbor relations between C and I consists of exactly $\lceil \frac{k}{3} \rceil$ copies of $C_{3,3}$ or $C_{2,2}$. More specifically, there are x copies of $C_{3,3}$ for $k = 3x$, there are $x-1$ copies of $C_{3,3}$ and two copies of $C_{2,2}$ for $k = 3x+1$, and there are x copies of $C_{3,3}$ and one copy of $C_{2,2}$ for $k = 3x+2$.

Suppose to the contrary that $\mathrm{BBC}_\lambda(G, M) \leq k+1$ for the case $k = 2m$, $m \geq 6$ and $\frac{k+6}{3} \leq \lambda \leq \frac{k}{2}$, or the case $k = 2m+1$, $m \geq 4$ and $\frac{k+6}{3} \leq \lambda \leq \frac{k+1}{2}$. Then the three following observations can be made.

Observation 4.5.5. *There is exactly one color that is not used in C, which we call the independent color in this case. Without loss of generality, we may assume that the independent color is in the set $\{\lambda+1, \ldots, k+1\}$. The independent color may be used p times in I, where $p \leq k+1-\lambda$. All vertices in I that are not colored with this independent color must get the same color as their unique nonneighbor in C, hence all these other colors can only occur once in I.*

Observation 4.5.6. *Assume that the independent color is in the set $\{\lambda+1, \ldots, k+1\}$ and that this color is used p times in I. Then we can choose only $k+1-\lambda-p$ colors from the set of other colors in $\{\lambda+1, \ldots, k+1\}$ to use them in I.*

Indeed, if the independent color is used $k+1-\lambda$ times, then all the possible colors for matching neighbors in C of the vertices in I with the other colors from $\{\lambda+1, \ldots, k+1\}$ are already in use by matching neighbors of the vertices that are colored with the independent color.

Observation 4.5.7. *Assume that the independent color is in the set $\{\lambda+1, \ldots, k+1\}$. Then the colors from $\{1, \ldots, \lambda\}$ can be used at most once in I. Even stronger, from the set $\{1, \ldots, \lambda\}$ we can choose only $\lceil \frac{k}{3} \rceil$ colors that can be used in I.*

Indeed, if we choose more, then there would be at least two colors from $\{1, \ldots, \lambda\}$ in one $C_{2,2}$ or $C_{3,3}$. This means that there would be a matching edge violating the minimally required distance λ between the two colors.

By these three observations, it can be derived that we can use the independent color at most p times in I, we can use the other colors from $\{\lambda+1, \ldots, k+1\}$ for at most $k+1-\lambda-p$ vertices of I, and we can use colors from $\{1, \ldots, \lambda\}$ for at most $\lceil \frac{k}{3} \rceil$ vertices of I. Since $\lceil \frac{k}{3} \rceil < \lambda - 1$, we can only color at most $k+1-\lambda+\lceil \frac{k}{3} \rceil < k$ vertices of I. We find a contradiction.

Finally, suppose to the contrary that $\text{BBC}_\lambda(G,M) \leq \frac{k+1}{2} + \lambda$ for the case $k = 2m+1$, $m \geq 4$ and $\lambda \geq \frac{k+1}{2} + 1$. It is clear that colors from the set $\{\frac{k+1}{2}+1, \ldots, \lambda\}$ can not be used at all. So, we can only use the $k+1$ colors from the two sets $\{1, \ldots, \frac{k+1}{2}\}$ and $\{\lambda+1, \ldots, \frac{k+1}{2}+\lambda\}$. Hence, we have one independent color. Without loss of generality, we may assume that this independent color is in $\{\lambda+1, \ldots, \frac{k+1}{2}+\lambda\}$. By Observation 4.5.5, we can use the independent color at most p times in I, where $p \leq \frac{k+1}{2}$. By Observation 4.5.6, we can use the other colors from $\{\lambda+1, \ldots, \frac{k+1}{2}+\lambda\}$ for at most the $\frac{k+1}{2} - p$ vertices of I. Since $\frac{k+1}{2} < \lambda$, by Observation 4.5.7, we can use colors from $\{1, \ldots, \frac{k+1}{2}\}$ for at most $\lceil \frac{k}{3} \rceil$ vertices of I. So we can only color at most $\frac{k+1}{2} + \lceil \frac{k}{3} \rceil$ vertices of I. However, since in this case $k \geq 9$, it holds that $\frac{k+1}{2} + \lceil \frac{k}{3} \rceil < k$. We find a contradiction. □

4.5.3 Tree backbones of split graphs

In this subsection we present sharp upper bounds for the λ-backbone coloring numbers of split graphs with tree backbones. The following theorem is a generalization of Theorem 1.4.4(a).

Theorem 4.5.8. *Let $\lambda \geq 2$ and let $G = (V,E)$ be a split graph with $\chi(G) = k$. For every tree backbone $T = (V, E_T)$ of G,*

$$\text{BBC}_\lambda(G,T) \leq \begin{cases} 1 & \text{if } k = 1 \\ 1+\lambda & \text{if } k = 2 \\ k+\lambda & \text{if } k \geq 3. \end{cases}$$

The bounds are tight.

Proof. We divide the proof into two parts as follows.

Part 1 Proof of the bounds.
Let $G = (V,E)$ be a split graph with a spanning tree $T = (V, E_T)$. Let C and I be a partition of V such that C with $|C| = k$ is a clique of maximum size, and such that I is an independent set. As before, $\chi(G) = \omega(G) = k$. The case $k = 1$ is trivial. If $k = 2$ then G is bipartite, and we use colors 1 and $\lambda+1$. For $k \geq 3$, we consider the restriction of the tree T to the vertices in C, and we distinguish two cases.

In the first case, the restriction of T to C forms a star $K_{1,k-1}$. Let $v_1, \ldots, v_{k-1}$ denote the $k-1$ leaves of this star, and let v_k denote its center. For $i =$

$1, \ldots, k-1$ we color v_i with color i, and we color v_k with color $k+\lambda-1$. This yields a λ-backbone coloring for the vertices in C. All vertices $u \in I$ are leaves in the tree T. Any vertex $u \in I$ with $uv_k \notin E_T$ can be safely colored with color $k+\lambda$. It remains to consider vertices $u \in I$ with $uv_k \in E_T$. In the graph G, such a vertex u is nonadjacent to at least one of the vertices $v_1, \ldots, v_{k-1}$, say to vertex v_j (otherwise, the clique C could be augmented by vertex u and would not be of maximum size as we assumed). In this case we may color u with color j.

In the second case, the restriction of T to C does not form a star. In this case the restriction of T to C has a proper 2-coloring $C = C_1 \cup C_2$ with $|C_1| = a \geq |C_2| = b \geq 2$. Then there exist a vertex $x \in C_1$ and a vertex $y \in C_2$ for which $xy \notin E_T$. Let $v_1, \ldots, v_a = x$ be an enumeration of the vertices in C_1, and let $y = v_{a+1}, \ldots, v_{a+b}$ be an enumeration of the vertices in C_2. For $i = 1, \ldots, a$ we color vertex v_i with color $i+1$. For $i = 1, \ldots, b$ we color vertex v_{a+i} with color $a+\lambda+i-1$. This yields a λ-backbone coloring of C with colors in $\{2, \ldots, k+\lambda-1\}$. We color each vertex $u \in I$ with color

$$\begin{cases} k+\lambda & \text{if } uv \in E_T \text{ and } v \in C_1 \\ 1 & \text{if } uv \in E_T \text{ and } v \in C_2. \end{cases}$$

This yields a λ-backbone $(k+\lambda)$-coloring of (G,T), since the colors of a vertex v_i with $i \in \{1, \ldots, a\}$ and of any vertex $u \in I$ such that $uv_i \in E_T$ have distance at least $k+\lambda-(i+1) \geq k+\lambda-(k-2+1) > \lambda$, and since the colors of a vertex v_i with $i \in \{a+1, \ldots, b\}$ and of any vertex $u \in I$ such that $uv_i \in E_T$ have distance at least $a+\lambda+i-1-1 \geq k/2+\lambda-1 \geq \lambda$.

Part 2 Proof of the tightness of the bounds.
The cases $k=1$ and $k=2$ are trivial. For $k \geq 3$, we consider a split graph with a clique of k vertices $v_1, \ldots, v_k$ and with an independent set of $(k-2)(k-1)/2$ vertices $u_{i,j}$ with $1 \leq i < j \leq k-1$. Every vertex $u_{i,j}$ is adjacent to all vertices v_s with $s \neq i$. The tree backbone T contains the $k-1$ edges $v_k v_s$ with $1 \leq s \leq k-1$. The vertices $u_{i,j}$ form the leaves of T; in the tree, vertex $u_{i,j}$ is adjacent only to v_j. Clearly, $\chi(G) = k$.

Suppose to the contrary that $\text{BBC}_\lambda(G,T) \leq k+\lambda-1$, and consider such a backbone coloring. The vertices $v_1, \ldots, v_k$ in the clique must be colored with k pairwise distinct colors. Since they form a star, either vertex v_k has color 1, and colors $2, \ldots, \lambda$ are not used on the clique, or vertex v_k has color $k+\lambda-1$, and colors $k, \ldots, k+\lambda-2$ are not used on the clique. Both cases are symmetric, and we assume without loss of generality that v_k has color $k+\lambda-1$ and that colors $k, \ldots, k+\lambda-2$ are not used on the clique. Let v_i be the vertex that has color $k-2$, and let v_j be the vertex that has color $k-1$. The vertex $u_{i,j}$ is adjacent to all clique vertices except v_i; hence, it could only be colored with color $k-2$ or with a color in $\{k,\ldots,k+\lambda-2\}$. But these λ colors are forbidden for $u_{i,j}$, since in the tree backbone it is adjacent to vertex v_j with color $k-1$. Since there is no feasible color for $u_{i,j}$, we arrive at the desired contradiction. □

4.6 The computational complexity of computing the λ-backbone coloring number

We consider the computational complexity of computing the λ-backbone coloring number: *"Given a graph G, a spanning subgraph H, and an integer ℓ, is* $\text{BBC}_\lambda(G,H) \leq \ell$*?"* Of course, this general problem is NP-complete. In this section we restrict ourselves to the graph G with a star backbone or a tree backbone. In Subsection 4.6.1 we show that for this problem the complexity jump occurs between $\ell = \lambda+1$ (easy for all star backbones S) and $\ell = \lambda+2$ (difficult even for matching backbones M). In Subsection 4.6.2 we show that for this problem the complexity jump occurs between $\ell = \lambda+2$ (easy for all tree backbones T) and $\ell = \lambda+3$ (difficult even for path backbones P).

4.6.1 Complexity results for star or matching backbones

Theorem 4.6.1. *Let* $\lambda \geq 2$.

(a) The following problem is polynomially solvable for any $\ell \leq \lambda+1$*: Given a graph G and a star backbone S, decide whether* $\text{BBC}_\lambda(G,S) \leq \ell$.

(b) The following problem is NP-complete for all $\ell \geq \lambda+2$*: Given a graph G and a matching backbone M, decide whether* $\text{BBC}_\lambda(G,M) \leq \ell$.

Proof. We start with the positive result in statement (a). So let $G=(V,E)$ be a graph with a star backbone $S=(V,E_S)$. For $\ell \leq \lambda$ the statement is trivial. Now let $\ell = \lambda+1$. We first note that in any λ-backbone coloring with color set $\{1,2,\ldots,\lambda+1\}$, colors $2,3,\ldots,\lambda$ can not be used at all, since each vertex is incident with an edge of E_S. Since the vertices with color 1 (color $\lambda+1$) form an independent set in G, it is clear that such a λ-backbone coloring induces a bipartition of G. On the other hand, if G is bipartite, then assigning color 1 and color $\lambda + 1$ to the vertices on both sides of the bipartition yields a λ-backbone coloring of any backbone of G. This shows that $\text{BBC}_\lambda(G, S) = \lambda + 1$ if and only if G is bipartite.

Now let us prove the negative result in statement (b). The reduction is done from the NP-complete classical problem of GRAPH k-COLORABILITY (see Garey & Johnson [19] problem [GT 4] for more information): Given a graph $H = (V_H, E_H)$, does there exist a k-coloring of H? This problem is known to be NP-complete for any fixed integer $k \geq 3$. We distinguish the following cases.

Case 1 $\lambda \geq 3$ and $\ell = \lambda + t$ for $t = 2, \ldots, \lambda - 1$.
Let $H = (V_H, E_H)$ be an instance of $2t$ colorability, and let $v_1, v_2, \ldots, v_n$ denote the vertices in V_H. We create n new vertices $u_1, u_2, \ldots, u_n$ and introduce the new edges $v_i u_i$ $(i = 1, 2, \ldots, n)$. The graph that results from this is denoted by G. The new edges form a matching backbone M of G. We claim that $\chi(H) \leq 2t$ if and only if $\text{BBC}_\lambda(G, M) \leq \ell$.

Assume that $\text{BBC}_\lambda(G, M) \leq l$ and consider a λ-backbone ℓ-coloring b of (G, M). Since all vertices in G are incident with a matching edge, colors $t + 1, t + 2, \ldots, \lambda$ can not be used at all. Then define a $2t$-coloring c of H by:

- if $b(v) = j$ for $j = 1, 2, \ldots, t$: $c(v) = j$;
- if $b(v) = \lambda + j$ for $j = 1, 2, \ldots, t$: $c(v) = t + j$.

Next, assume that $\chi(H) \leq 2t$, and consider a $2t$-coloring $f : V_H \to \{1, , \ldots, 2t\}$. We define a λ-backbone ℓ-coloring $g : V_G \to \{1, \ldots, \ell\}$ of (G, M) by:

- if $v \in V_H$ and $f(v) = j$ for $j = 1, 2, \ldots, t$: $g(v) = j$;
- if $v \in V_H$ and $f(v) = t + j$ for $j = 1, 2, \ldots, t$: $g(v) = \lambda + j$;
- if $g(v_i) \leq t$: $g(u_i) = \ell$;
- If $g(v_i) \geq \lambda + 1$: $g(u_i) = 1$.

Case 2 $\lambda \geq 2$ and $\ell \geq 2\lambda$.
Let $H = (V_H, E_H)$ be an instance of ℓ colorability, and let $v_1, v_2, \ldots, v_n$ denote the vertices in V_H. We create n new vertices $u_1, u_2, \ldots, u_n$ and introduce new

edges $v_i u_i$ ($i = 1, 2, \dots, n$). The graph that results from this is denoted by G. The new edges form a matching backbone M of G. We complete the proof by showing that $\chi(H) \le \ell$ if and only if $\text{BBC}_\lambda(G, M) \le \ell$.

Indeed, assume that $\text{BBC}_\lambda(G, M) \le \ell$ and consider such a λ-backbone ℓ-coloring. Then the restriction to the vertices in V_H yields an ℓ-coloring of H. Next assume that $\chi(H) \le \ell$, and consider an ℓ-coloring $f : V_H \to \{1, , \dots, \ell\}$. We extend f to a λ-backbone ℓ-coloring of (G, M): If $f(v_i) \le \lambda$, then vertex u_i is colored with color ℓ, and otherwise it is colored with color 1. This completes the proof. □

4.6.2 Complexity results for tree or path backbones

Theorem 4.6.2. *Let $\lambda \ge 2$.*

(a) The following problem is polynomially solvable for any $\ell \le \lambda + 2$: Given a graph G and a spanning tree T, decide whether $\text{BBC}_\lambda(G, T) \le \ell$.

(b) The following problem is NP-complete for all $\ell \ge \lambda + 3$: Given a graph G and a Hamiltonian path P, decide whether $\text{BBC}_\lambda(G, P) \le \ell$.

Proof. We start with the positive result in statement (a). Let $G = (V, E)$ be a graph with a spanning tree $T = (V, E_T)$. The cases where $\ell \le \lambda$ are trivial. Now let $\ell = \lambda + 1$ and $V = V_0 \cup V_1$ be the bipartition of the vertex set induced by T. Then in any λ-backbone coloring with color set $\{1, \dots, \lambda + 1\}$, colors $2, \dots, \lambda$ can not be used at all. Consider some fixed vertex $v \in V_0$. Without loss of generality assume that the color of v is 1. Then all vertices in V_0 must be colored with 1, and all vertices in V_1 must be colored with $\lambda + 1$. Hence, $\text{BBC}_\lambda(G, T) = \lambda + 1$ if and only if G is bipartite.

Next, consider the case of a λ-backbone coloring with color set $\{1, \dots, \lambda + 2\}$. Then colors $3, \dots, \lambda$ can not be used at all. Consider some fixed vertex $v \in V_0$. Without loss of generality assume that the color of v is in $\{1, 2\}$. Then all vertices in V_0 must be colored with colors in $\{1, 2\}$, and all vertices in V_1 must be colored with colors in $\{\lambda + 1, \lambda + 2\}$. Hence, $\text{BBC}_\lambda(G, T) \le \lambda + 2$ if and only if the two subgraphs of G that are induced by V_0 and by V_1 are both bipartite with the additional condition that none of the edges of E_T has end vertices with color 2 in V_0 and color $\lambda + 1$ in V_1. Checking these conditions can be modeled as a 2-SAT problem, as follows. We introduce two Boolean variables

x_v and y_v for each vertex $v \in V(G)$, where we let the two literals x_v and $\overline{x}_v$ correspond to assigning color 1 or color 2 to v, respectively, and y_v and $\overline{y}_v$ to assigning color $\lambda + 1$ or color $\lambda + 2$ to v, respectively. Now $G[V_0]$ is bipartite if and only if there is a satisfying truth assignment for $(x_u \vee x_v) \wedge (\overline{x}_u \vee \overline{x}_v)$ for each edge $uv \in E(G[V_0])$. A similar statement holds for $G[V_1]$. Finally, an edge $uv \in E_T$ with $u \in V_0$ is properly colored according to a λ-backbone $\lambda+2$-coloring if and only if there is a satisfying truth assignment for $x_u \vee \overline{y}_v$. Since 2-SAT is polynomially solvable (see Garey & Johnson [19]), this completes the proof of the statement in (a).

Now let us prove the negative result in statement (b) of Theorem 4.6.2. The reduction is done from the NP-complete classical problem of GRAPH k-COLORABILITY. We distinguish the following cases.

Case 1 $\ell = \lambda + t$ for $t = 3, \dots, \lambda$.
Let $H = (V_H, E_H)$ be an instance of t colorability, and let $v_1, v_2, \dots, v_n$ be an enumeration of the vertices in V_H. We create $n-1$ new vertices $a_1, a_2, \dots, a_{n-1}$. For every $i = 1, \dots, n-1$ we introduce the new edges $v_i a_i$ and $a_i v_{i+1}$. The graph that results from adding these $n-1$ new vertices and these $2(n-1)$ new edges to H is denoted by G. The vertices $v_1, a_1, v_2, a_2, v_3 \dots, a_{n-1}, v_n$ form a Hamiltonian path $P = (V_P, E_P)$ of G. We complete the proof by showing that $\chi(H) \le t$ if and only if $\text{BBC}_\lambda(G, P) \le \ell$.

Assume that $\text{BBC}_\lambda(G, P) \le \ell$ and consider a λ-backbone ℓ-coloring b of (G, P). Since $t \le \lambda$, in any λ-backbone coloring only colors in $\{1, \dots, t\} \cup \{\lambda+1, \dots, \lambda+t\}$ can be used. Note that $V = V_H \cup \{a_1, \dots, a_{n-1}\}$ is the bipartition of the vertex set induced by P. Consider some fixed vertex $v \in V_H$. Without loss of generality assume that the color of v is in $\{1, \dots, t\}$. Then all vertices in V_H must be colored with colors in $\{1, \dots, t\}$. Hence $\chi(H) \le t$.

Next, assume that $\chi(H) \le t$, and consider a t-coloring $f : V_H \to \{1, , \dots, t\}$. We extend f to a λ-backbone ℓ-coloring of (G, P): Every vertex a_i receives color $\lambda + t$.

Case 2 $\ell \ge 2\lambda + 1$.
Let $H = (V_H, E_H)$ be an instance of ℓ colorability, and let $v_1, v_2, \dots, v_n$ be an enumeration of the vertices in V_H. We create $3(n-1)$ new vertices a_i, b_i, c_i with $1 \le i \le n-1$. For every $i = 1, \dots, n-1$ we introduce the new edges $v_i a_i$, $a_i b_i$, $b_i c_i$, and $c_i v_{i+1}$. The graph that results from adding these $3(n-1)$ new vertices and these $4(n-1)$ new edges to H is denoted by G. The vertices $v_1, a_1, b_1, c_1, v_2, a_2, b_2, \dots, c_{n-1}, v_n$ form a Hamiltonian path P of G. We claim that $\chi(H) \le \ell$ if and only if $\text{BBC}_\lambda(G, P) \le \ell$.

Indeed, assume that $\text{BBC}_\lambda(G,P) \leq \ell$ and consider such a λ-backbone ℓ-coloring. Then the restriction to the vertices in V_H yields a proper ℓ-coloring of H. Next assume that $\chi(H) \leq \ell$, and consider a ℓ-coloring $f : V_H \to \{1,,\ldots,\ell\}$. We extend f to a λ-backbone ℓ-coloring of (G,P):

- Every vertex b_i receives color $\lambda + 1$.
- If $f(v_i) \leq \lambda + 1$, then a_i is colored ℓ, and otherwise it is colored 1.
- If $f(v_{i+1}) \leq \lambda + 1$, then c_i is colored ℓ, and otherwise it is colored 1.

This completes the proof of Theorem 4.6.2. □

Summary

In this thesis we consider the following three topics in graph theory: spanning 2-connected subgraphs of grid graphs, Ramsey numbers for paths versus other graphs, and some variations of vertex colorings.

In Chapter 1 we present some notations and give an overview of the main results obtained, together with a survey of related known results.

In Chapter 2 we define some classes of grid graphs that we call truncated rectangular grid graphs and alphabet graphs. We solve the problem of determining a spanning 2-connected subgraph with as few edges as possible for these graphs.

In Chapter 3 we determine the Ramsey numbers for paths versus wheels $R(P_n, W_m)$, the Ramsey numbers for paths versus kipases $R(P_n, \hat{K}_m)$ and the Ramsey numbers for paths versus fans $R(P_n, F_m)$ for some values of m and n. We also give lower bounds and upper bounds for $R(P_n, W_m)$, $R(P_n, \hat{K}_m)$ and $R(P_n, F_m)$ for the other values of m and n.

In Chapter 4 we study combinatorial and algorithmic aspects of λ-backbone colorings. We determine a relation between the chromatic numbers and the λ-backbone coloring numbers of graphs with star backbones or matching backbones. We also consider the special cases where the graph is a planar graph and the backbone is a matching, and where the graph is a split graph and the backbone is a collection of pairwise disjoint stars or a perfect matching or a tree. Finally, we study the computational complexity of λ-backbone coloring for a graph with a star backbone, with a matching backbone, with a tree backbone or with a path backbone.

Samenvatting

In dit proefschrift beschouwen wij de volgende drie onderwerpen uit de grafentheorie: opspannende 2-samenhangende deelgrafen van roostergrafen, Ramsey-getallen voor paden versus andere grafen, en enige varianten van puntkleuringen.

In Hoofdstuk 1 geven wij enige notaties alsmede een overzicht van de belangrijkste behaalde resultaten, samen met een overzicht van verwante bekende resultaten.

In Hoofdstuk 2 definiëren wij enige klassen van roostergrafen die wij afgeknotte rechthoekige roostergrafen en alfabetgrafen noemen. Wij lossen het probleem op een opspannende 2-samenhangende deelgraaf met zo weinig mogelijk lijnen te bepalen voor deze grafen.

In Hoofdstuk 3 bepalen wij de Ramsey-getallen voor paden versus wielen $R(P_n, W_m)$, de Ramsey-getallen voor paden versus kipassen $R(P_n, \hat{K}_m)$ en de Ramsey-getallen voor paden versus waaiers $R(P_n, F_m)$ voor enige waarden van m en n. Wij geven ook ondergrenzen en bovengrenzen voor $R(P_n, W_m)$, $R(P_n, \hat{K}_m)$ en $R(P_n, F_m)$ voor de overige waarden van m en n.

In Hoofdstuk 4 bestuderen wij combinatorische en algorithmische aspecten van λ-skeletkleuringen. Wij bepalen een verband tussen de chromatische getallen en de λ-skeletkleuringsgetallen van grafen met sterskeletten of matchingskeletten. Wij beschouwen eveneens de speciale gevallen, waarin de graaf planair is en het skelet een matching, en waarin de graaf een split-graaf en het skelet een kollektie van paarsgewijs disjunkte sterren is of een perfekte matching, dan wel een boom. Tenslotte bestuderen wij de berekeningskomplexiteit van λ-skeletkleuring voor een graaf met als skelet een ster, een matching, een boom of een pad.

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About the Author

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